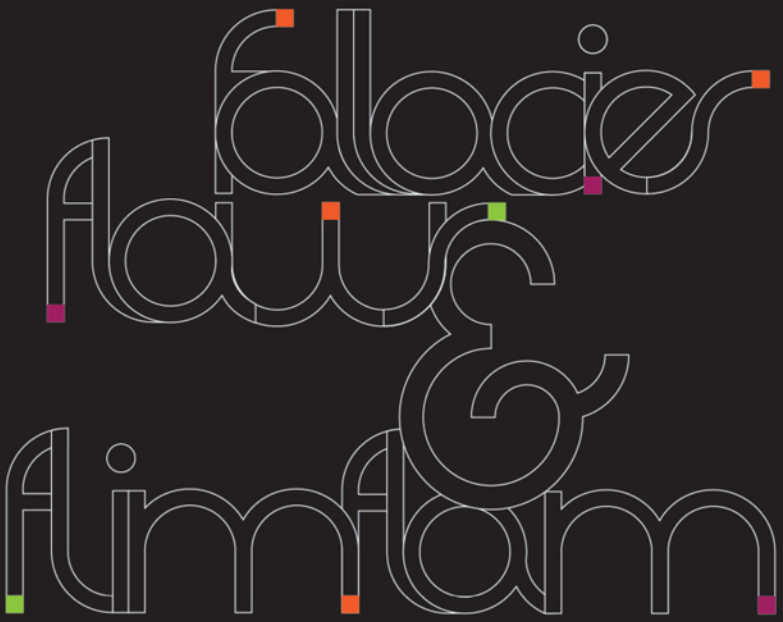
More Fallacies, Flaws & Flimflam



Edward J. Barbeau

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More Fallacies, Flaws, and Flimflam

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Library of Congress Catalog Card Number 2013950210

Print edition ISBN: 978-0-88385-580-5

Electronic edition ISBN: 978-1-61444-515-9

Printed in the United States of America

Current Printing (last digit):
10 9 8 7 6 5 4 3 2 1

More Fallacies, Flaws, and Flimflam

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Published and Distributed by
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Preface

Here is another collection of *Fallacies, Flaws and Flimflam*, mostly drawn from the column of this name in the *College Mathematics Journal* between 2000 and 2008. As in the first volume, there is a variety of items ranging from howlers (outlandish procedures that nonetheless lead to a correct answer) to errors that are deep or subtle often made by strong students. While some are provided for entertainment, others offer a challenge to the reader to determine exactly where things go wrong.

While there are many proposals to improve the quality of mathematics education, these seldom address the need for students to pay careful attention to what they do and to check their work. It is through an engagement with meaning that they can avoid the pitfalls that come too naturally. Accordingly, my hope is that this volume might be useful to teachers at all levels. It may be that the task of sorting out erroneous solutions may endow students with empathy for the hapless teacher who is faced with such solutions all the time and has to come up with a constructive response.

The items are sorted according to subject matter. Elementary teachers will not find much of use beyond Chapter 1, while middle and secondary teachers will find items in Chapters 1, 2, 3, 7, 8 that they might use. College teachers should find material in every part of the book. There are frequent references to the *College Mathematics Journal*; these are denoted by *CMJ*.

I am grateful to both the contributors who provided the material and the reviewers of the manuscript whose careful reading and constructive criticism led to some corrections of errors and improvements. I would like to thank Beverly Ruedi for her careful and attractive typesetting of the text. Finally, I give tribute to my wife, Eileen, for her patience and support throughout this whole enterprise.

Ed Barbeau

CHAPTER 1

Arithmetic

1.1 Percentages

Many mathematical missteps in the public sphere involve percentages. It is a good habit to note explicitly the quantity to which a percentage is applied, as a common error is to add or subtract percentages applied to different quantities.

For example, suppose a woman earned a salary of $40,000 last year and got a raise of 10% this year, but expects a salary cut of 10% next year. This will not leave her where she started. The first 10% is applied to $40,000, so her earnings this year are $44,000. It is this amount to which the 10% reduction is applied, so that next year, the salary will be reduced by $4400 to the amount $39,600, for a net reduction of 1% from last year's salary.

Example 1

Mr. and Mrs. Buyer are visiting the ClothesLine Shop to get Mr. Buyer a coat. A suitable item has been found; the regular price is $160, but this is reduced by 25% because of the spring sale. However, a sales tax of 15% applies to the actual cost of the item.

The sales clerk proposes adding the 15% sales tax to the original price of $160, and then take 25% off the resultant amount to get the cost to the customer. Mrs. Buyer is suspicious. "Hold on," says she. "I do not see why the tax should be worked out on the original price. I think I am getting ripped off. You should take the 25% off the price first, and then add the 15% sales tax to that."

Mr. Buyer, becoming impatient with all of this disputation, finally blurts out, "You both are making a big deal out of this. Look, there is 15% that

has to be added in and 25% that has to be taken off, so just take 10% off the price and we will be square." □

Who is right? Thanks to the associativity and commutativity of multiplication, the sales clerk and the wife arrive at the same amount:

$$\begin{aligned}(160 \times 1.15) \times 0.75 &= 160 \times (1.15 \times 0.75) \\ &= 160 \times (0.75 \times 1.15) \\ &= (160 \times 0.75) \times 1.15.\end{aligned}$$

This gives the correct amount, \$138, which is the sum of the sale price, \$120, and the tax of \$18 on this amount. The husband, however, errs, as 15% and 25% are not applied to the same amounts and so the latter cannot simply be subtracted from the former. Taking 10% off the price amounts to charging the sales tax of \$24 on \$160, even though the item costs only \$120.

Example 2

Problem Last year, Meredith got a raise of 10% in her salary. Unfortunately, there was 20% inflation, so that it now costs \$1.20 to buy what a dollar would have purchased a year ago. So, in effect, Meredith suffered a pay decrease of a certain percentage. What was that percentage?

Solution (by a student) Meredith's current salary is 11/10 of her salary last year. However, the value of a current dollar is 10/12 of a dollar a year ago. Hence, her salary this year is really worth $(11/10)\times(10/12) = (11/12)$ of her salary last year. Let Meredith's salary be \$10,000. She would receive $(11/12) \times (10000) = 9167$ [last year's] dollars this year for her salary (base plus raise) because of the inflation.

From this, we can determine the percent pay decrease that she suffered. But first, we must determine the percentage of her last year's salary that she made this year by taking the amount she made this year and dividing this by the amount she made last year and multiplying the resultant amount by 100%.

Therefore, the percentage of Meredith's last year's salary that she made this year is $(9167/11000) \times 100 = 83.34$. Therefore, this year Meredith made 83.34% of what she made last year. She suffered a pay decrease of 16.66%. □

The solution is on firm ground until the end of the first paragraph, where her current salary of \$11,000 is worth \$9167 in last year's dollars. This

drop from \$10,000 to \$9167 represents a decrease of about 8.33% in buying power. The student erred by taking the ratio both of whose parts are this year's value: its value in last year's dollars and its value in this year's dollars. The amount she earned last year plays no role in the computation; this is clearly wrong because a comparison of the purchasing power of both year's salaries is being asked for.

An alternative perspective would be that, if her salary had increased at the rate of inflation, she would get \$12,000 this year. Instead, she obtained only \$11,000, an effective reduction of \$1000, which is 8.33% of \$12,000.

Example 3

In CMJ 42:4 (September, 2011), Christopher Thron considers a problem, drawn from a textbook, to find the present value of \$100 three years from now, assuming that money can be invested at 4% per year and that inflation is 3% per year. Without accounting for inflation, the present value of \$100 would be $\$100/(1.04)^3 = \$88.90$. If we look at buying power, \$100 three years from now will purchase what $\$100/(1.03)^3 = \$91.51$ does today.

Thron examines and rejects two possible solutions for the combined present value of \$100. In the first one, since both interest and inflation reduce the present value, the answer should be $(\$100 \div (1.03)^3) \div (1.04)^3 = \$81.36$. In the second one, he notes "if I have \$88.90 now, I bank it and withdraw \$100 after three years. When I spend it, I buy stuff costing \$91.51 at today's prices. So it seems that \$88.90 now is worth \$91.51 in three years. At this rate, \$100 in three years has a present value of \$97.14. Strange, it seems inflation has actually *increased* the present value of \$100 (since it was only \$88.90 without inflation)!" After discarding both of these answers, he concludes that there is "no meaningful way to combine them taking both inflation and interest rates into account."

However, this goes against the intuition that we should be able to make a comparison. Look at matters this way. The \$88.90 that, invested, will grow to \$100 in three years will be less than the \$91.51 worth of goods that will cost \$100 in three years. So the present value of \$100, taking account of both interest and inflation, is $\$(88.90/91.51) \times 100 = \$97.15$. It is not surprising that this exceeds \$88.90, as we have to invest a larger sum to counteract the loss of buying power due to inflation.

1.2 Loss of face

I am indebted to Richard Askey for drawing my attention to the following exercise in troy measure found in a widely used algebra text authored by a

team of fourteen individuals. The exercise is one of several in a group with the heading "Applying the Mathematics." The answer is from the teacher's edition.

Exercise

Helen of Troy was described as having "a face that could launch a thousand ships." If a face could launch a thousand ships, what would it take to launch five ships?

Answer. 0.005 face.

I suppose a wrinkle of the nose would just about do it. It is hard to fathom the demeanor of the author who put this question in the book. Is there whimsy here? Or was it done in all seriousness? Perhaps the intent was to demonstrate that mathematics really does connect outside of itself, as there is a sidebar to the question:

> **History Connection.** According to Greek legend, Helen of Troy was the most beautiful woman of Greece and the indirect cause of the Trojan War. She was taken to Troy by Paris, the son of the Trojan king, even though she was the wife of Menelaus, the king of Sparta. Menelaus sent a large number of ships to capture Troy and bring Helen back to Greece. The phrase "the face that launched a thousand ships" is from the poem "The Tragical History of Doctor Faustus" by the English poet Christopher Marlowe.

David Cantrell responds: "Is there whimsy here? Surely! Apparently, you are unaware of the joke, long a favorite of mine, which is well known, especially in the scientific community. Just do a *Google* search for 'millihelen'. You will find many references. From one of them, I extracted the following:

> *Helen* (H) is huge unit of beauty, and is thus inconvenient to work with. The smaller unit, the milliHelen (mH), representing the amount of beauty to launch a single ship is more workable."

1.3 Syllabus innumeracy and the easy A

Charles Redmond writes:

> "Quiz 1 is 10% of your final grade." We've all seen statements like this in a syllabus. Many professors write out their grading schemes this way, followed by letter grade ranges such as "90–100 A." I know

this, because I've gotten several frantic phone calls from colleagues in other departments who did write their syllabi this way and couldn't figure out how to calculate their final grades at the end of the term. There's a good reason. They said something they didn't mean to say, and they had only a vague notion of what they meant to say in the first place. If you are a student, this can be turned to your advantage. Suppose quiz 1 is worth 10 or more points, and you get a score of 9. Then let x be your final grade. Since "Quiz 1 is 10% of your final grade," solve $0.10x = 9$ to get 90 as your final grade and an A standing. Not bad for about a week's worth of work. Take the rest of the semester off.

1.4 Addition by juxtaposition

Brendan Kelly gave the following howler to his cadet teachers to analyse. Suppose we wish to add two fractions with the same single-digit denominator and with single-digit numerators, say $(2/7) + (4/7)$. First of all we juxtapose the denominators to get 77. Then we juxtapose the numerators. But the sum should be independent of the order of adding the fractions, so we should juxtapose the numerators both ways and add them together:

$$\frac{2}{7} + \frac{4}{7} = \frac{24 + 42}{77} = \frac{66}{77} = \frac{6}{7}.$$

It works every time.

1.5 Toothpicks

In a Grade 7 test item for pupils learning about algebraic expressions, Elaine Simmt of the University of Alberta in Edmonton discovered that some children were getting the right answer with the wrong formula. A 3×6 rectangle of unit squares can be constructed from 45 toothpicks serving as sides of unit squares. The pupils surmised that an $m \times n$ rectangle would need $m^2 + n^2$ toothpicks, which turns out to be correct when $(m, n) = (3, 6)$. To challenge this, Simmt suggested looking at other examples and had the misfortune to pick $(m, n) = (6, 10)$.

It is not hard to see which pairs (m, n) fail to provide a counterexample. We require that

$$m^2 + n^2 = (m + 1)n + (n + 1)m = 2mn + m + n,$$

which reduces to $(m-n)^2 = m+n$. Taking the parameter $t = m-n$, we are led to

$$(m,n) = \left(\frac{t(t+1)}{2}, \frac{t(t-1)}{2}\right),$$

a pair of triangular numbers.

1.6 A multiplicity of multiplications

Teachers need to be aware that a student may offer an answer that is non-standard, but is in fact correct and can be justified. Here is a case in point. Hyman Bass, in the followup to his talk at ICME-10 in Copenhagen, Denmark, presented three possible ways in which pupils might prepare a multiplication computation:

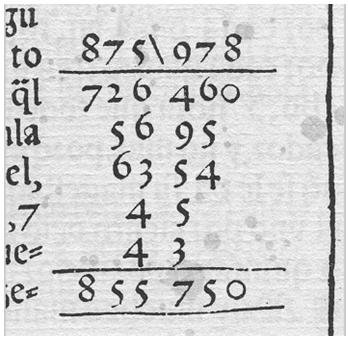
$$\begin{array}{rrr} & 3 & 5 \\ \times & 2 & 5 \\ \hline \\ 1 & 2 & 5 \\ 7 & 5 & \\ \hline \\ 8 & 7 & 5 \end{array} \qquad \begin{array}{rrr} & 3 & 5 \\ \times & 2 & 5 \\ \hline \\ 1 & 7 & 5 \\ 7 & 0 & 0 \\ \hline \\ 8 & 7 & 5 \end{array} \qquad \begin{array}{rrr} & 3 & 5 \\ \times & 2 & 5 \\ \hline \\ & 2 & 5 \\ 1 & 5 & 0 \\ 1 & 0 & 0 \\ \hline 6 & 0 & 0 \\ \\ 8 & 7 & 5 \end{array}$$

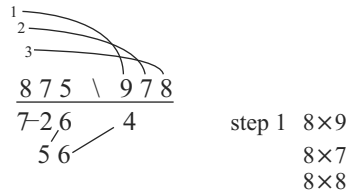
Why do the methods work?

This item drew a very interesting response from Shirley B. Gray. She states that the first multiplication algorithm to be printed in the Western Hemisphere appeared in the *Sumario Compendioso*, published in Mexico City in 1556. The author, Brother Juan Diez, brought his European education to the New World.

The algorithm, written in Spanish, instructs readers to start on the left and move to the right, *i.e.*, the opposite of what is taught today. First look at the complete calculation of 875×978 as it appears in the text (image on the left below).

Now, follow Brother Juan's instructions: "Multiply by 8 from left to right and build below." Diez then wrote: "Now leave the 8 and take the 7 for the multiplier $\cdots$ " Spanish students will understand "ahora ver a el, 8, y tomo el 7 $\cdots$." The 7 products start to build one place value moved to the right and below. The last multiplier is, of course, the 5 in the ones (or 10^0) place. Note that Diez does not "carry" and uses only two digit products, His





algorithm is valid for one digit products if the column alignment is thought of as 0#; thus, $3 \times 3 = 09$.

There are four known surviving copies of the *Sumario*: at the Huntington Library, the British Museum, Duke University and Salamanca, Spain. The facsimile copy of the Salamanca book, reprinted in 1985 to honor the visit of the President of Mexico to Madrid, is available in several libraries. Others may have a copy in translation of David Eugene Smith's *The Earliest Mathematical Work of the New World*, Ginn, 1921. This collection of algorithms and 6 pages of algebra problems make a wonderful group project for a History of Mathematics course.

Vaughan Pratt notes that there is an interesting point to be made about Shirley Gray's statement that the first multiplication algorithm to be printed in the Western Hemisphere is in Brother Juan Diez' 1556 *Sumario Compendioso*. "If one believes everything one reads in *Wikipedia*, there was a time when Oxford University was part of the Western Hemisphere (see the yellow map in the Wikipedia entry of that name). Robert Recorde's first chapter on multiplication, in his 1542 book, *Ye Grounde of Artes — Teachyng the worke and practise of arithmetike* was written while Recorde was a fellow of All Souls College, Oxford, 1.26 degrees west of Greenwich. (Cambridge, located at 0.12 degrees East, in Wikipedia's judgment has always belonged to the Eastern Hemisphere, where Recorde studied medicine starting in 1545.)" Pratt adds that Recorde later became controller of the Royal Mint, but died in a debtors' prison. Pratt sent me a PDF copy of the chapter on multiplication from his copy of the 1582 edition. While one cannot believe everything in Wikipedia, he was glad that it gave him a pretext to mention Recorde's contribution.

Pratt remarks that "the chapter gets off to a brisk start with the following cute algorithm for multiplying two digits, a and b, each in the range 5 to 9."

Find $(10-a)\times(10-b)$ and then add to it 10 times the last digit of $a+b$. This reduction to the easy products in the range 1 to 5 exploits the identity

$$(10-a)(10-b)+10(a+b)=100+ab.$$

A detailed biography of Recorde can be found at `www.history.mcs.st-andrews.ac.uk/Biographies/Recorde.html`.

Yves Nievergelt points out that Hyman Bass's and Juan Diez's algorithms for multiplication [4] (cited by Shirley B. Gray [6]) are similar to algorithms implemented in digital computers (as opposed to their contemporary *analog* computers [1]) from the 1950's through today.

The IBM 650 multiplied two 10-digit decimal integers — called *multiplier* and *multiplicand* — with a 20-digit *accumulator* [7, p. 35]. The multiplicand was placed in a *distributor* (D) and the multiplier in the *upper* half (U) (10 most significant digits) of the accumulator while the *lower* half (L) (10 least significant digits) was set to zero.

The entire (20-digit) accumulator was shifted by one digit to its left, filling its least significant digit with a zero, and ejecting the *most* significant digit of the multiplier into a register (R). While that register's content was strictly positive, the register iteratively subtracted 1 from its content and added the multiplicand (D) to the lower half of the accumulator, in effect multiplying — through repeated additions — the *most* significant digit of the multiplier by the multiplicand and adding the product to the current total. With three instead of ten digits, the IBM 650 multiplies Diez' example $875*978$ as follows:

						D			
					9	7	8		
	0	8	7	5	0	0	0		
	R		U			L			
shift	8	7	5	0	**0**	**0**	**0**	$0*978$	
	7	7	5	0	**9**	**7**	**8**	$1*978$	
	6	7	5	**1**	**9**	**5**	**6**	$2*978$	
	5	7	5	**2**	**9**	**3**	**4**	$3*978$	
	⋮	⋮	⋮	⋮	⋮	⋮	⋮		
	1	7	5	**6**	**8**	**4**	**6**	$7*978$	
	0	7	5	**7**	**8**	**2**	**4**	$8*978$	
shift	7	5	**7**	**8**	**2**	**4**	**0**	$80*978$	
	6	5	**7**	**9**	**2**	**1**	**8**	$81*978$	
	⋮	⋮	⋮	⋮	⋮	⋮	⋮		
result	0	**8**	**5**	**5**	**7**	**5**	**0**	$875*978$	

With copies in operation in Austria, Belgium, Finland, France, Greece, Turkey, Sweden, and West Germany, the IBM 650 is the most frequently mentioned computer in Nelson B. Blachman's survey of computers in Europe in 1960 [2].

DEUCE used a nearly identical algorithm to multiply two 32-bit binary integers in a 64-bit *long accumulator* [5, p. 61–66]. Binary representations simplify the multiplication somewhat. The current most significant digit of the multiplier is either 1, in which case the multiplicand is added once, or 0, in which case the multiplicand is not added and the whole accumulator is shifted for the next most significant digit. However, the accumulator was shifted to its *right*, because the representation listed the *least* significant digits first (at left) and the *most* significant digits last (at right) [5, p. 29], as for infinite series of positive powers. The distinction is purely academic, because reading the manual or looking at the machine upside down reverses the order of the digits.

Modern computers compute all partial products of one digit of the multiplier by one digit of the multiplicand at once in parallel, exactly as in Diez, and Hyman Bass's third example, and then add all these partial products in a cascading tree of adders [3]. This so-called *Gelosia* method had already appeared 900 years ago in India [3].

References

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[2] Nelson M. Blachman, The state of the digital computer technology in Europe, *Communications of the ACM* 4:6 (June, 1961) 256–265.

[3] Peter R. Capello and Kenneth Steiglitz, A VLSI layout for a pipelined Dadda multiplier, *ACM Transactions on Computer Systems* 1 (1983) 157–174.

[4] Juan Diez, *Sumario compendioso de las quentas de plata y oro que in los reynos del Piru son necessarias a los mercaderes y todo genero de tratantes. Los algunas reglas tocantes al Arithmetica.* A printing press in Mexico City, 1556.

[5] English Electric, *Logical manual*, 1957. `users.tpg.com.au/eedeuce/`

[6] Shirley B. Gray and C. Edward Sandifer, The sumario compendioso: the New World's first mathematics book, *Mathematics Teacher* 94 (2001) 98.

[7] International Business Machines Corporation, *IBM 650 Magnetic-Drum Data Processing Machine Manual of Operation* (revised), 1955. `bitsavers.org/pdf/ibm/650/22-6060-2_650_OperMan.pdf`

1.7 Hot stuff in Canada

In April, 2005, Ontario experienced unusually warm weather. Neal Madras sent the following explanation, published on page B1 of the April 20, 2005 edition of the *Toronto Star*.

> **Why it's been so warm.** The jet stream moved up over Northern Ontario and warm southerly air moved in. Storms have been blocked and diverted by the intense high pressure area. By the weekend, it will be south of us and Arctic air will cause the temperature to drop.
>
> **52% warmer this month**
> April 1–18 average temp. **7.9 C**
> Normal average **5.2 C**

1.8 Which sex philanders more?

The following item appeared in the *Atlantic Monthly* for March, 2010, in the Letters Column on page 17.

> MISLEADING MATH?
>
> Megan McArdle attempts to illustrate her point that survey respondents are unreliable ("Misleading Indicator," November [2009] *Atlantic*) by telling us that it is mathematically impossible for men to report an average number of female sexual partners that is much higher than the average number of male partners reported by women. I agree that survey respondents are unreliable, but so is McArdle's math. It may be unlikely that the number of partners reported by honest males would be higher, but it is not mathematically impossible.
>
> Fred Graf, Concord, NH

Megan McArdle replies:

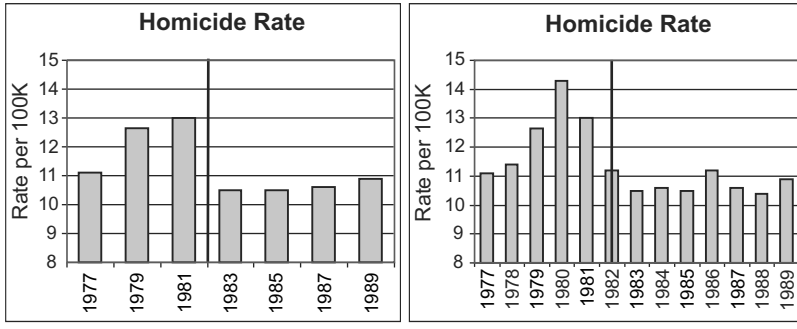
> We are talking about two different meanings of the word average. True, in Harvey Levin's scenario, it is more common for women to be monogamous than men. But I was using average to describe the mean number of sexual partners, which is to say, the sum of everyone's number of partners, divided by the number of people. If there are 10 men and 10 women, and one of the women has slept with all 10 men, while the other women are monogamous, the average number of sexual partners is the same for the men and the women. Nine of the men have had two sexual partners, while one of the men has had

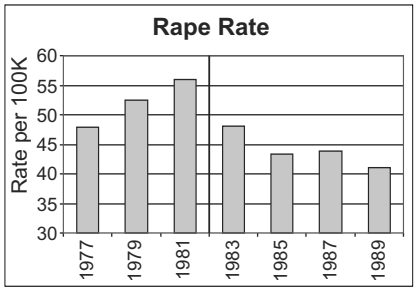
one partner, for an average of $19/10 = 1.9$. Meanwhile, one of the women has had 10 sexual partners while the other nine women have had one partner, for an average of $19/10 = 1.9$. The numbers come out the same, no matter how you vary the particulars. Yet surveys can generate differences of three- or fourfold between the mean numbers of sexual partners that women and men report.

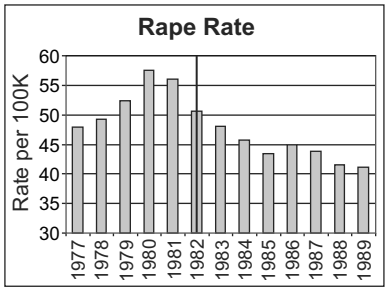
1.9 Proposition 8 in California

In 1982, the state of California enacted, by referendum, Proposition 8, that provided longer prison terms for repeat offenders of the crimes of murder, rape, robbery, aggravated assault with a firearm and burglary of a residence. The deterrent effects of this law were examined in a 1999 article by Daniel Kessler and Stephen Levitt [**1**], who looked at statistics for both eligible and noneligible crimes for the legislation prior to and following its passage. Taking into account many factors, they concluded the the legislation was effective in reducing the rate of eligible offenses by about 4%.

This conclusion was challenged by Cheryl Marie Webster, Anthony Doob and Franklin Zimring in 2009 [2]; Levitt responded in the same issue [3]. The arguments and counterarguments are too detailed and technical to be recounted here, and readers interested in the issue should consult the papers themselves. However, a significant criticism made by the 2009 authors was that Kessler and Levitt used only statistics from the odd numbered years to make their case. Webster, Doob and Zimring presented the data in the form of histograms for each of the five eligible offenses; the graphs below are for homicide and rape. The graphs on the left use the data of Kessler and Levitt, while those on the right fill in for the even years. As the reader can see, the inclusion of the additional data makes the case for the deterrent effect of the proposition considerably weaker.







References

[1] Daniel Kessler & Steven D. Levitt, Using sentence enhancement to distinguish between deterrence and incapacitation. *Journal of Law and Economics* 42 (1999), 343–363.

[2] Cheryl Marie Webster, Anthony N. Doob & Franklin E. Zimring, Proposition 8 and crime rates in California: the case of the disappearing deterrence. *Criminology & Public Policy* 5:3 (2006), 417–448.

[3] Steven D. Levitt, The case of the critics who missed the point: a reply to Webster *et al. ibid.* 5:3 (2006), 449–460.

1.10 A skill-testing question

In the March, 2005 issue of the *Ontario Mathematics Gazette* (Vol. 43, No. 3), Jack Weiner of the University of Guelph in Ontario reported receiving a letter from a mother advocating on behalf of her son. In Ontario, it is not legal for one to win a commercial draw unless one answers a "skill testing question." This regulation dates back to more puritanical times in the history of the province. When the law was first passed, the question was often a bit of historical or social trivia, but the almost universal practice now is that it be arithmetical. This particular lad won second prize in a draw whose first prize was a *Honda Nighthawk*; he was to get $200 in compact discs. To claim his prize, he had to answer the following question: *What is* $22 + 14 \div 3 \times 4 - 3$*?*

He submitted the answer $37\frac{2}{3}$ and was informed by Honda that he could not get the prize as the answer was incorrect. "You see," he was told, "$22 + 14$ is 36, which when divided by 3 is 12. Multiply this by 4 to get 48. Finally, subtract 3 to arrive at the correct answer of 45."

Weiner suggests that one might give the following example to illustrate the *BEDMAS* issue at stake. "You have $100 saved and are working part-time at McDonald's. Say the amount is about $40 [that you can save each

week]. Now everyone will agree that the student will, after ten weeks, have $\$100 + \$40 \times 10 = \$500$ saved. However, according to Honda's arithmetic, the student would have $\$140 \times 10 = \$1400$. Hmm, I thought. Maybe we should use Honda's method."

Weiner wrote back to the mother supporting the student's answer. A colleague suggested that, as his time was valuable, he should charge a consultation fee. So he attached to the letter the sentence, "My consultation fee is $\$1000 - \$500 \times 2$." He later phoned the mother to follow up and asked about the fee. "With a smile in her voice, she replied, 'The cheque is in the mail'."

1.11 Computing the cost of a fence

Johnny Lott got the following solution to a problem from Georgia Cobbs. A pupil was asked to compute the cost of fencing the perimeter of a rectangular field 132 feet long and 99 feet wide, given that there are 16.5 feet in a rod and one rod of fencing costs $12.75.

The pupil first performed five multiplications:

$$165 \times 4 = 660$$
$$165 \times 7 = 1155$$
$$165 \times 6 = 990$$
$$165 \times 9 = 1485$$
$$165 \times 8 = 1320$$

All but the third and fifth of these were crossed out, and the student continued with two more multiplications and two additions:

$$1275 \times 6 = 7650$$
$$1275 \times 8 = 10200$$
$$10200 + 7650 = 17850$$
$$17850 + 17850 = 35700$$

The last sum was stated to be the answer. Indeed the required cost is $357.

1.12 The dream of Homer Simpson

In an article in the *San Francisco Chronicle* dated December 16, 2005, Dick Rogers discusses the disposition of a correction to a previous article. On

October 15, a staff writer, Steve Rubenstein, described how the three writers for "The Simpsons" television show wove mathematical references into the series. In particular, in one episode, Homer Simpson, "in a dream, wrote that 1782 to the 12th power plus 1841 to the 12th power equals 1922 to the 12th power." Unfortunately, Rubenstein followed this with the parenthetical comment, "It does."

A parity check reveals the falsity of the result, and reactions from readers made the issuing of a correction necessary. Rogers was told by the city desk that it was not enough to say the equation was false; without the proper answer, "the correction doesn't correct." With the help of Sam Brannen, a mathematician at Sonoma State University at Rohnert Park, CA, using *Mathematica*, the paper was eventually able to print:

> A story [...] mistakenly said that 1782 to the 12th power plus 1841 to the 12th power equals 1922 to the 12th power. Actually, 1782 to the 12th power plus 1841 to the 12th power equals
>
> 2,541,210,258,614,589,176,288,669,958,142,428,526,657
>
> while 1922 to the 12th power equals
>
> 2,541,210,259,314,801,410,819,278,649,643,651,567,616.

One wonders whether Rubenstein checked the equation with a pocket calculator that showed a limited number of significant digits.

1.13 Increasing a square to a square

Chris Fisher thought that readers would be interested in the following solution given by one of his students to a problem from the 1987 Hungarian National Olympiad (reproduced in *Crux Mathematicorum* 17:3 (March, 1991), 70).

Problem. N is a four-digit square all of whose digits are less than seven. Increasing each digit by three, we obtain a perfect square again. Find N.

Solution. Let the square root of our number be $10a + b$ (where a and b are digits). The given condition tells us that

$$(10a + b + 33)^2 = (10a + b)^2 + 3333$$

so that

$$(10a + b)^2 + 2 \times 33 \times (10a + b) + 33^2 = (10a + b)^2 + 3333.$$

Cancel the $(10a + b)^2$ from each side and divide by 33; then $2(10a + b) = 68$, so $a = 3$ and $b = 4$. □

Indeed, $34^2 = 1156$ and $67^2 = 4489$. Is this a fluke? Or is it a valid approach that shows remarkable insight? Suppose that $N = u^2$ and that the larger square is $w^2 = u^2 + 3333$. Then

$$101 \times 3 \times 11 = w^2 - u^2 = (w - u)(w + u).$$

The student's solution corresponds to the particular factorization

$$(w - u)(w + u) = 33 \times 101.$$

However, we would have to check other possible factorizations of 3333 as a two-fold product.

1.14 Attributed to Vladimir Putin

I am indebted to Andre Toom for drawing my attention to a quotation attributed to the Russian leader, Vladimir Putin. It can be found on the website `anekdot.ru/an/an0606/y060626.html`, in the section "Best stories of the past years." According to Toom, the "stories pretend to be real and avoid impossible details," but "you can never exclude the possibility that they are true." Here is the entry for June 26, 2002, translated from the Russian by Toom:

> Yesterday, V.V. Putin, answering questions of journalists, said: ... The Belorussian economy equals 3% of the Russian, and the Russian, accordingly, 97% of the Belorussian.
>
> What a mathematician ...

1.15 An odd ending

When Sung Soo Kim was in fifth grade of elementary school, his teacher tried to stop pupils from tearing pages out of their notebooks for paperfolding by having them number the pages. When he found that one student had numbered the last page with an odd number, he remarked to the lad that surely there was an error in the numbering. However, the lad did not believe him until the teacher went through checking the book with him page by page.

1.16 Lighter than air

A colleague found the following problem and solution in the teachers' edition of *Geometry: a modern approach* by Marie S. Wilcox, published in 1968 by Addison-Wesley. This was printed in the margin as a "twister."

Problem. Mr. Flatell decided that his wife had to go on a diet. He drew up a weight losing program in which she was to lose 3.1 pounds the first month, 4.7 pounds the second month, 16.8 pounds the third month, and 39.3 pounds the fourth month. If Mrs. Flatell weighed 260 pounds at the start of this program, how much would she weigh at the end of five months?

Solution. The weight to be lost each month is calculated by finding one-tenth of the square of the sum of the digits of the weight lost the previous month and adding that quantity to the previous month's weight loss. For example, for the second month,

$$(1/10)(3+1)^2 = 1.6\,; \quad 1.6 + 3.1 = 4.7.$$

Thus, in the fifth month Mrs. Flatell should lose 61.8 pounds. She would then weigh 134.3 pounds. □

It is not specified whether Mrs. Flatell remained on the diet for another couple of months. This problem raises some issues. For example, how would this diet operate in a country on the metric system? What would happen if Mr. Flatell were more meticulous in specifying the initial weight loss to more than one place after the decimal point?

Since this item originally appeared in the *College Mathematics Journal*, an internet search has brought to light an old dispatch from the *Canadian Press*:

> Swastika, ON (Feb. 13, 1969, CP) The local detachment of the Ontario Provincial Police has recently detained Mr. Graham Flatell on suspicion of having murdered his wife, Eloise Flatell. She disappeared in January of this year, after, according to her husband, having gone to Churchill, Manitoba, to follow her hobby of polar bear watching. It was initially believed that she was stranded with a polar bear and subsequently devoured when the ice floe upon which she was stationed detached from the shore.
>
> However, the suspicions of the police were aroused when it was learned that Mr. Flatell had urged his wife to go on a diet last July, and her friends became concerned when she began to lose ever larger

amounts of weight each month. Mrs. Rena Snurd, a neighbor, reported that by the end of December, Mrs. Flatell was “a mere shadow of her former self” and that in the ensuing weeks, she seemed “to vanish into thin air.” When no evidence of a trip to Churchill was found, police began to look into the possibility of foul play. No body has been recovered to date.

[Reader alert: This, of course, is pure fiction.]

CHAPTER 2

School Algebra

2.1 "Very funny, Peter"

For some light relief, we begin with this item which has been circulating in the ether as the scan of a pupil's script for many years. It appears to be genuine. The title of this item was written by the teacher, along with an "X," beside the following solution:

Problem Expand $(a+b)^n$.

$$(a+b)^n = (a\ +\ b)^n = (a\ \ +\ \ b)^n = (a\ \ \ +\ \ \ b)^n = (a\ \ \ \ +$$

2.2 A sum of squares surprise

Li Zhou sends the following problem, which appeared on a sample group examination for the Florida Math Olympics:

Problem If $x > 0$ and $x + \frac{1}{x} = 1$, then evaluate $x^2 + \frac{1}{x^2}$.

This is a mathematics contest for Florida community college students. It is usually held in April at the University of North Florida in Jacksonville, and consists of three parts: 40 multiple-choice questions (1 hour, 30 minutes); 10 calculator problems (45 minutes); 10 group problems (2 hours, 30 minutes). Zhou comments that "Olympics questions are indeed very tricky!"

2.3 An algebra problem

This problem was set on an algebra test:

Problem Solve for x:

$$\frac{5(2x-6)}{7} = \frac{3(4x-12)}{2}.$$

It is reported that one student multiplied both sides by 2 and forgot the 7, a second multiplied both sides by 7 and forgot the 2, and a third neglected both the 7 and the 2. All got correct answers.

It is interesting that the question was set at all. Was it set deliberately, or was it a fluke that both sides had the factor $(x-3)$? As we have seen, all sorts of wrong manipulations would lead to a correct answer. However, if any students did notice the common factor $(x-3)$, it is not clear that they would know how to handle the situation.

2.4 The escaped criminal

Recently, I gave a number of secondary students a problem that appeared in *The High School Algebra* by Robertson and Birchard, approved for use in Ontario schools in 1886. I am indebted to the no longer extant *Secondary School Mathematics Bulletin* (from the University of Waterloo, ON) for relaying this problem. While many students had difficulty setting the problem up correctly, I was surprised to see how often they nevertheless obtained the correct answer. It turns out that some of the information is redundant.

Problem A criminal, having escaped from prison, traveled for 10 hours before his escape was detected. He was then pursued and gained upon at 3 miles per hour. When his pursuers had been 8 hours on the way, they met an express going in the opposite direction at the same rate as themselves, which had met the criminal 2 hours and 24 minutes earlier. In what time from the beginning of the pursuit will the criminal be overtaken?

The information that the pursuers travel 3 miles per hour faster than the criminal is redundant. Let u be the speed in miles per hour of the criminal and v be the speed in miles per hour of the pursuers and of the freight train. The distance in miles from the prison to the place where the freight train encountered the criminal is $8v + (2.4)v = (18 - 2.4)u$, which implies that $2v = 3u$. After 8 hours of pursuit, the distance between the criminal and his pursuers is

$$18u - 8v = t(v-u)$$

for some value of t. This equation reduces to

$$(18+t)u = (t+8)v,$$

which with the earlier equation yields $t = 12$. Thus after 8 hours, the distance between the criminal and his pursuers is $18u - 8v = 12(v - u)$. Since the pursuers are traveling at speed $v - u$ relative to the criminal, each passing hour reduces the distance between criminal and pursuers by $v - u$. Thus, it will take them 12 more hours to close the gap. Hence it will take 20 hours after the pursuers begin to catch the criminal.

2.5 A collection of howlers

One does not require much teaching experience before encountering solutions by pupils who mangle standard algebraic rules, yet somehow end up with the correct answer. Even though the mathematics is completely wrong, it often has its own internal consistency.

(a) This howler was brought to my attention by Ross Honsberger; it seems remarkable that it actually delivers the correct result.

Exercise Simplify

$$\frac{(a+b)^3 + a^3}{(a+b)^3 + b^3}.$$

Solution Cancel the 3s to get $(2a + b)/(a + 2b)$. □

(b) Dale Buske provides more examples of the student notion that all functions are linear. To the problem of solving the equation

$$\log x - \log 3 = \log 5 - \log(x - 2),$$

one student deduced the statement $x - 3 = 5 - (x - 2)$ resulting in $x = 5$, while a second offered this compelling argument:

$$\begin{aligned}
\log x - \log 3 &= \log 5 - \log x - \log(-2) \\
2\log x - \log 3 &= \log 5 - \log(-2) \\
2\log x - \log 3 &= \log 5 + \log 2 \\
2\log x - \log 3 &= \log 7 \\
2\log x &= \log 10 \\
x &= 5. \ \square
\end{aligned}$$

In fact, for any positive a and b, similar machinations will yield a correct solution to the equation

$$\log x - \log a = \log(a + b) - \log(x - b).$$

(c) Here is an item from Carl Libis. A student verifies the sum of a geometric series and makes remarkably consistent use of a new way of handling exponential functions.

Problem Use the Principle of Mathematical Induction to prove that

$$1 + 3 + 9 + 27 + \cdots + 3^n = (3^{n+1} - 1)/2$$

for all $n \geq 0$.

Solution The formula holds for $n = 0$. Assume that it holds for $n = k$. Then

$$\begin{aligned}
&1 + 3 + 9 + 27 + \cdots + 3^k + 3^{k+1} \\
&\quad = (1 + 3 + 9 + 27 + \cdots + 3^k) + 3^{k+1} \\
&\quad = [(3^{k+1} - 1)/2] + 3^{k+1} = \frac{3^k + 3^1 - 1}{2} + 3^k + 3^1 \\
&\quad = \frac{3^k + 2}{2} + \frac{6^k + 6}{2} = \frac{9^k + 8 + (1 - 1)}{2} = \frac{9^k + 9 - 1}{2} \\
&\quad = \frac{3(3^k + 3) - 1}{2} = \frac{3(3^{k+1}) - 1}{2} = \frac{3^{k+2} - 1}{2}.
\end{aligned}$$

The result now follows.

(d) J. Sriskandarajah posed this problem on his Spring 2000 final examination in College Trigonometry. The solution here is an amalgam of two student solutions.

Problem Prove

$$\frac{\sin^2 x - \cos^2 x}{\sin^2 x - \sin x \cos x} = 1 + \cot x.$$

Solution The left side is equal to

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin x \cos x} = 1 + \frac{\cos x}{\sin x} = 1 + \cot x.$$

(e) Problem Prove the identity

$$\sin^2 A - \sin^2 B = \sin(A + B) \sin(A - B).$$

Solution

$$\begin{aligned}\sin^2 A - \sin^2 B &= \sin(A^2 - B^2) = \sin[(A + B)(A - B)] \\ &= \sin(A + B)\sin(A - B).\end{aligned}$$

Anand Kumar discovered this proof on a test script from a first year college student. Some will already be familiar with other "derivations" of this identity. (See *CMJ* 20 (1989) 404, FFF #11; 22 (1991) 132–133).)

(f) Problem Solve

$$\frac{1}{x+1} + \frac{1}{x-2} = \frac{x+3}{x^2 - x - 2}.$$

Solution (by a student in a college algebra course).

$$\begin{aligned}\frac{1}{x+1} + \frac{1}{x-2} &= \frac{x+3}{x^2 - x - 2} \\ \frac{x+1}{1} + \frac{x-2}{1} &= \frac{x^2 - x - 2}{x+3} \\ x + 1 + x - 2 &= \frac{x^2 - \frac{x}{x} - 2}{3} \\ 2x - 1 &= \frac{x^2 - 1 - 2}{3} \\ 3(2x - 1) &= x^2 - 3 \\ 6x - 1 &= x^2 - 3 \\ 6x &= x^2 - 2 \\ \frac{6x}{2x} &= \frac{x^2 - 2}{2x} \\ 3 &= \frac{x^2}{x} - \frac{2}{2} \\ 3 &= x - 1 \\ 4 &= x.\end{aligned}$$

Contributed by Carl Libis.

(g) Problem Prove that, if $d_k = 2^k - 1$ for all k, then $d_k = 3d_{k-1} - 2d_{k-2}$.

Solution

$$\begin{aligned} 3d_{k-1} - 2d_{k-2} &= 3(2^{k-1} - 1) - 2(2^{k-2} - 1) \\ &= \frac{3(2^k)}{2} - 3 - \frac{2(2^k)}{2^2} + 2 \\ &= \frac{6^k}{2} - \frac{4^k}{4} - 1 = 3^k - 1^k - 1 \\ &= 2^k - 1 = d_k. \end{aligned}$$

Contributed by Carl Libis.

(h) Problem Prove that $4^{2n} - 2^{2n}$ is divisible by 6 for each positive integer n.

Solution

$$4^{2n} - 2^{2n} = 16^n - 4^n = 8 \cdot 2^n - 2 \cdot 2^n = 6 \cdot 2^n = 12^n.$$

Since 12 is divisible by 6, so is any power of 12, and the proof is complete.

Contributed by Parviz Khalili.

(i) Carl Libis presented a student solution for the equation

$$4^{x-1} = 2^x.$$

This equation is equivalent to

$$4(x-1) = 2x \Leftrightarrow 4x - 4 = 2x \Leftrightarrow 2x = 4 \Leftrightarrow x = 2.$$

The equation $4^{x-1} = 2^x$ is an example of a situation where $a^c = b^d$ and $a^b = b^a$ hold simultaneously. In this case, $ac = bd$ and it is not surprising that the student's solution leads to the correct answer. In general, if both the equations $a^c = b^d$ and $ac = bd$ hold, then we require that $b \log a = a \log b$ or $a^b = b^a$. When $a \neq b$ and we set $a = b^u$, then the condition $a^b = b^a$ leads to $bu = b^u$ or $(a, b) = (u^{u/(u-1)}, u^{1/(u-1)})$. Conversely, such a pair (a, b) satisfies $a^b = b^a$. (It is a nice exercise for students to change the roles of a and b, set $b = a^v$ and then verify that the substitution $v = 1/u$ gives the foregoing formulae.)

2.6 Dimensions of a yard

Problem The distance around a rectangular yard is 228 feet. If the length of the yard is 6 feet more than the width, find the dimensions of the yard.

Solution (by a student) Divide 228 by 2 to get 114. Knowing that the length is 6 feet longer than the width, I subtracted 6 from 114 which gives me 108 and add the 6 I subtracted onto the length which makes 120. I then divided the 108 by 2 to get the result which is 54 feet long. By dividing 120 by 2, I would get 60 which is the same as the length. □

This method actually works in general, although any justification I can think of seems rather contrived. For example, 114 is the sum of the length and width, and so is twice the average of these dimensions. Similarly, 6, the difference of the dimensions, must be twice the deviation of them from the average. So, if we increase and decrease the sum of the dimensions by 6, we must get twice their values. It would be interesting to know how the student providing this solution framed it in his/her mind.

2.7 A faulty reasoning approach to solving a quadratic

A problem that appeared in the *International Mathematical Talent Search* created and circulated by George Berszenyi notes that, for each quadratic equation $(x - p)(x - q) = 0$, there exist constants a, b, c with $c \neq 0$ for which the equation is equivalent to $(x - a)(b - x) = c$ and the false reasoning "either $x - a = c$ or $b - x = c$ yields the correct answers '$x = p$' or '$x = q$'." For example, the equation $(x - 19)(x - 97) = 0$ can be rewritten as $(x - 18)(98 - x) = 79$.

Since the two quadratics are equivalent, equating coefficients yields $p + q = a + b$ and $pq = ab + c$. Suppose that $q = a + c$ and $p = b - c$ are the two roots of the equation. Then

$$ab + c = pq = ab + c(b - a - c),$$

whence, since $c \neq 0$, $c = b - a - 1$. Then $p = b - c = a + 1$ and $q = a + c = b - 1$. Thus,

$$(a, b, c) = (p - 1, q + 1, q - p + 1)$$

and $(x-p)(x-q) = 0$ is equivalent to $(x-p+1)(q+1-x) = q-p+1$.

2.8 The illegal moves method for quadratics

A more subtle, fishy-looking but ultimately justifiable method of solving equations was drawn to my attention by John C. and Holly M. Hoover.

They pointed out that pupils in beginning algebra are taught four methods of solution for quadratic equations: factoring, taking square roots, completing the square and using the quadratic formula. Most of them find difficult the factorization of quadratics for which the leading coefficient is not equal to 1. In an effort to relieve some of the anxiety her students were feeling, one intermediate algebra teacher searched the internet for approaches alternative to those in texts. She found one: the "Illegal moves method." This is easily memorized by students and seems to lighten the burden of teaching the factorization of quadratic trinomials. However, the morality of introducing this method and recommending its use is questionable. For, the illegal moves method seems to support the idea that "the end justifies the means" by requiring the user to commit three algebra sins to arrive at the result.

What is it? Suppose that $ax^2 + bx + c$ has to be factored, where a, b, c are integers with no common factors save 1, a is neither 0 nor 1, and the discriminant $b^2 - 4ac$ is the square of an integer, so that the roots of the quadratic are rational. Follow these steps:

1. Remove the coefficient of the x^2 term by multiplying it into the constant term. This produces $x^2 + bx + ac$.
2. Now factor $x^2 + bx + ac$. Since this factors over the integers, there exist integers b_1 and b_2 for which $x^2 + bx + ac = (x + b_1)(x + b_2)$.
3. Next, divide each of the constants by a: $(x + b_1/a)(x + b_2/a)$.
4. Reduce the fractions to lowest terms:
$$(x + b_1/a)(x + b_2/a) = (x + u_1/v_1)(x + u_2/v_2).$$
5. "Squeeze" the denominator of each fraction in front of the binomials to get the desired factorization: $(v_1x + u_1)(v_2x + u_2)$.

Here is how the method goes on an example: factor $6x^2 - 7x - 3$.

1. Remove the 6 and multiply it into the constant term: $x^2 - 7x - 18$.
2. Factor the new trinomial: $(x - 9)(x + 2)$.
3. Divide each constant by 6: $(x - \frac{9}{6})(x + \frac{2}{6})$.
4. Reduce fractions to lowest terms: $(x - \frac{3}{2})(x + \frac{1}{3})$.
5. Finally, move each denominator to the leading term: $(2x - 3)(3x + 1)$, which is the factored form of $6x^2 - 7x - 3$.

This factorization method for quadratics drew comments from three readers, Phil Wood, Ioana Mihaila and David Bloom. They thought that the method was very good, and Wood in particular regretted the title ("the illegal moves method") which unnecessarily put it under a cloud.

James C. Alexander points out that both the equations $ax^2 + bx + c = 0$ and $x^2 + bx + ac = 0$ yield the same numerator in the quadratic formula for their roots so that the roots of the former are equal to $1/a$ times the roots of the latter; from this, it is easy to see how a factorization of the second quadratic induces a factorization of the first. It may be that this comparison of the roots of the two quadratic equations led to the promulgation of the method in the first place. However, one can still question the wisdom of introducing students to such an opaque method, even if it is efficient.

2.9 Continuing a sequence

A nice problem in the International Mathematical Talent Search asks for the value of $ax^4 + by^4$ when it is given that

$$\begin{aligned} a + b &= 23 \\ ax + by &= 79 \\ ax^2 + by^2 &= 217 \\ ax^3 + by^3 &= 691. \end{aligned}$$

One way to approach the solution is to derive from each pair of successive equations the relations $b(y - x) = 79 - 23x$, $by(y - x) = 217 - 79x$, $by^2(y - x) = 691 - 217x$. Therefore

$$\begin{aligned} y &= \frac{217 - 79x}{79 - 23x} = \frac{651 - 237x}{237 - 69x} \\ &= \frac{691 - 217x}{217 - 79x} = \frac{40 + 20x}{-20 - 10x} = -2. \end{aligned}$$

Doing the same maneuver to isolate the value of x leads to $x = -2$. However, it is straightforward to see that $(x, y) = (-2, -2)$ cannot satisfy the equation (otherwise the numbers on the right side would be in geometric progression).

To take another approach, one could note that the numbers x and y are the roots of a quadratic equation of the form $t^2 = ut + v$, so that

$$217 = 79u + 23v$$

and

$$691 = 217u + 79v.$$

This system has the solution $(u, v) = (1, 6)$ and we find that $ax^4 + by^4 = 691 + 6 \times 217 = 1993$. We can go ahead to obtain the values of all the variables:

$$(a, b; x, y) = (-2, 25; -2, 3), (25, -2; 3, -2).$$

Why did the first method not turn up these solutions?

The problem, as is often the case, is dealing with a fraction for which the denominator can possibly vanish. The rational function $(40 + 20x)/(-20 - 10x)$ is indeterminate when $x = -2$. However, the other rational functions appearing in the expression for y are well-defined and take the value 3 when $x = -2$. And indeed we find that $(x, y) = (-2, 3)$ appears as part of the solution of the system. Observe that when $x = 3$, all of the rational functions yield the value -2 and we get $(x, y) = (3, -2)$. When x assumes some other value, then the rational functions assume different values and the equations for y do not hold. A similar phenomenon occurs when we solve for x in terms of y.

2.10 The surreptitious "solution"

Omicron: I am trying to solve a system of equations proposed by Stanley Rabinowitz in the Spring, 1982 issue of *AMATYC Review*:

$$\begin{aligned} x + xy + xyz &= 12 \\ y + yz + yzx &= 21 \\ z + zx + zxy &= 30. \end{aligned}$$

Upsilon: Let's see. If we let $u = xyz$, then $x + xy = 12 - u$ so that $z + z(12 - u) = 30$ and $z = 30/(13 - u)$. Similarly, from the second and third equations, $y = 21/(31 - u)$ and from the first two equations, $x = 12/(22 - u)$. Plugging these expressions into any one of the three equations yields that

$$0 = u^3 - 65u^2 + 1306u - 7560 = (u - 10)(u - 27)(u - 28).$$

We get the three solutions

$$(x, y, z) = (1, 1, 10), \left(-\frac{12}{5}, \frac{21}{4}, -\frac{15}{7}\right), (-2, 7, -2),$$

all of which satisfy the system.

Omicron: It seems to me that it would be simpler to substitute the expressions for x, y, z into the equation $xyz = u$. This leads to the quartic equation

$$\begin{aligned} 0 &= u(u-13)(u-22)(u-31) + (12)(21)(30) \\ &= u^4 - 66u^3 + 1371u^2 - 8866u + 7560 \\ &= (u-1)(u^3 - 65u^2 + 1306u - 7560). \end{aligned}$$

Apart from the three values of u already identified, we have $u = 1$. This leads to

$$(x, y, z) = \left(\frac{4}{7}, \frac{7}{10}, \frac{5}{2}\right).$$

While, indeed, $xyz = 1$, we find that $x + xy + xyz = 69/35$, $y + yz + xyz = 69/20$, $z + zx + xyz = 69/14$. Where did this maverick "solution" come from and what is its significance?

Upsilon: I think it may be related to the fact that each of the values of x, y, z in terms of u arises from a different pair of the three equations. For example, we use the first and third equations to find that $z = 30/(13-u)$. If we stick with these two equations, we can find that $xy = u/z = u(13-u)/30$, whereupon

$$x = (12-u) - xy = (12-u) - \frac{u(13-u)}{30} = \frac{u^2 - 43u + 360}{30}$$

and

$$y = \frac{u(13-u)}{u^2 - 43u + 360}.$$

The triple

$$(x, y, z) = \left(\frac{u^2 - 43u + 360}{30}, \frac{u(13-u)}{u^2 - 43u + 360}, \frac{30}{13-u}\right)$$

gives a single infinity of solutions to the first and third equations. It will satisfy the second if and only if u satisfies the cubic equation $0 = (u-10)(u-27)(u-28)$.

Omicron: This same cubic equation arises as the condition that the value of y in this parametrization is equal to $21/(31-u)$ or that the value of x is equal to $12/(22-u)$.

Upsilon: The solution $x = 12/(22-u)$ is consistent with and comes from the first two equations, the solution $y = 21/(31-u)$ from the second and third, and the solution $z = 30/(13-u)$ from the first and third. Unless

u satisfies the cubic equation, these values for x, y, z are not consistent with all three equations. Using the equation $xyz = u$ to solve for u avoids the three equations together. In fact, this leads to

$$12 \times 21 \times 30 = u(13-u)(22-u)(31-u)$$

which is clearly satisfied by $u = 1$.

Omicron: If we generalize the three equations, replacing 12, 21, 30 by a, b, c respectively, we see that the same phenomenon occurs. When $u = 1$, we find that $(x, y, z) = (a/b, b/c, c/a)$ and that $x + xy + xyz = va$, $y + yz + yzx = vb$, $z + zx + zxy = vc$, where $v = a^{-1} + b^{-1} + c^{-1}$. Thus, if the sum of the reciprocals of a, b, c is 1, then $u = 1$ actually does correspond to a solution of the three equations.

Upsilon: In the general situation, with $u = xyz$, we find that u must satisfy the cubic equation

$$\begin{aligned} 0 &= u^3 - (a+b+c+2)u^2 + (ab+bc+ca+a+b+c+1)u - abc \\ &= u(u-1)[u-(a+b+c+1)] + (ab+bc+ca)u - abc. \end{aligned}$$

We see that $u = 1$ satisfies this equation if and only if $ab+bc+ca = abc$.

2.11 A trigonometric equation

Prove that $\tan 20^\circ + 4 \sin 20^\circ = \sqrt{3}$.

Solution Let $a = \cos 20^\circ$. Then

$$\frac{1}{2} = \cos 60^\circ = 4a^3 - 3a$$

and

$$\frac{\sqrt{3}}{2} = \sin 60^\circ = \sin 20^\circ (4a^2 - 1).$$

The equation is equivalent to

$$\tan 20^\circ + 4 \sin 20^\circ = 2 \sin 20^\circ (4 \cos^2 20^\circ - 1).$$

Dividing by $\sin 20^\circ$, we obtain that $(1/a) + 4 = 8a^2 - 2$ so that $1 = 8a^3 - 6a$. Since this is true, the required equation holds. □

This solution is ingenious, but there is a subtle difficulty with it. All the ingredients are present, but they need a shuffle. Basically, the argument in

the solution is that the desired equation implies a true statement and so must be true itself. Let us make the logic flow in the correct direction:

$$\begin{aligned} 1 = 8a^3 - 6a &\Longrightarrow \frac{1}{a} + 4 = 8a^2 - 2 \\ &\Longrightarrow \frac{1}{\cos 20^\circ} + 4 = 8\cos^2 20^\circ - 2 \\ &\Longrightarrow \tan 20^\circ + 4\sin 20^\circ = 2\sin 20^\circ(4\cos^2 20^\circ - 1) = \sqrt{3}. \end{aligned}$$

2.12 Summing squares by averages

The following question was posed on a test following the completion of a unit on arithmetic and geometric progressions. The solution was provided by a student.

Problem Let A and B be quantities defined thus:

$$\begin{aligned} A &= 2^2 + 4^2 + 6^2 + \cdots + 98^2 + 100^2, \\ B &= 1^2 + 3^2 + 5^2 + \cdots + 97^2 + 99^2. \end{aligned}$$

Find the difference between A and B.

Solution We use the well-known formula

$$\text{sum} \;=\; \text{number of terms} \;\times \left(\frac{\text{first term} \;+\; \text{last term}}{2} \right).$$

Applying the formula, we get

$$A = 50 \times \left(\frac{2^2 + 100^2}{2} \right), \qquad B = 50 \times \left(\frac{1^2 + 99^2}{2} \right),$$

and so

$$A - B = 25 \times (2^2 + 100^2 - 1^2 - 99^2) = 25 \times (3 + 199) = 5050. \qquad \square$$

This was submitted by Shailesh Shirali, who notes that, not only is the answer correct, but the method works when the first n odd squares are subtracted from the first n even squares for all positive integers n.

2.13 A constant logarithm sum

Proposition *Let a, b be positive reals exceeding* 1. *Then the function* $\log_a x + \log_b x$ *is constant for* $x > 1$.

Solution We have that

$$\begin{aligned}\log_a x + \log_b x &= \frac{1}{\log_x a} + \frac{1}{\log_x b} \\ &= \frac{\log_x a + \log_x b}{\log_x a \log_x b} \\ &= \frac{\log_x (ab)}{\log_x a \log_x b} \\ &= \left[\frac{\log_x a \log_x b}{\log_x (ab)}\right]^{-1} \\ &= [\log_{ab} a \log_{ab} b]^{-1} = \log_a (ab) \log_b (ab) \\ &= 2 + \log_a b + \log_b a\end{aligned}$$

which is independent of x. □

The mistake in this one is hard to pick out; it occurs at the fifth equals sign. Actually, the quantity in the square brackets is

$$\left(\frac{\log_x a}{\log_x (ab)}\right) \log_x b = \log_{ab} a \log_x b$$

which is not independent of x.

2.14 An exponential equation

Problem Solve for real values of x,

$$3^x \cdot 8^{x/(x+2)} = 6.$$

Solution An obvious solution is $x = 1$. Consider the function $f(x) = 3^x 8^{x/(x+2)}$. Since x and $x/(x+2) = 1 - [2/(x+2)]$ are strictly increasing, so is $f(x)$. Hence $f(x) = 6$ has a single solution, namely $x = 1$. □

Comment Note, however, that when $x = -2 \log_3 6$, then $3^x = 6^{-2}$ and

$$\frac{x}{x+2} = \frac{-2 \log_3 6}{-2 \log_3 6 + 2} = \frac{\log_3 6}{\log_3 6 - 1} = \frac{\log_3 6}{\log_3 2} = \log_2 6$$

so that $8^{x/(x+2)} = 6^3$. Thus, $x = -2 \log_3 6$ satisfies the equation.

Henry J. Barten generalizes to

$$3^x 8^{x/(x+2)} = k$$

for positive parameter k. Since the graph of the left side increases from 0 to infinity on each of the half lines $(-\infty, -2)$, $(-2, \infty)$, the equation has two real solutions. When $k = 6$, these are 1 and $-\log_3 36$, and when $k = 1$, these are 0 and $-\log_3 72$. By taking logarithms to base 3, we see that the equation is equivalent to

$$x^2 + [\log_3(72/k)]x - \log_3(k^2) = 0.$$

This has two distinct real solutions for $k > 0$, although in general their expressions are not very palatable.

2.15 Logging the solutions of an equation

Problem Solve for real x the equation

$$2^{x+2}5^{6-x} = 10^{x^2}.$$

Solution 1 The equation has one solution. It can be rewritten as

$$2^{x+2}5^{6-x} = 2^{x^2}5^{x^2}.$$

Therefore $x + 2 = x^2$ and $6 - x = x^2$. These quadratic equations have one common solution $x = 2$, which satisfies the given equation. □

Solution 2 The equation has one solution. The equation can be rewritten as $f(x) = 1$ where

$$\begin{aligned} f(x) &= 2^{(x^2-x-2)}5^{(x^2+x-6)} \\ &= 2^{(x+1)(x-2)}5^{(x+3)(x-2)} \\ &= \left[2^{(x+1)}5^{(x+3)}\right]^{(x-2)}. \end{aligned}$$

Observe that $f(2) = 1$.

We show that $f(x)$ is strictly increasing. Suppose that $u < v$. Then $2^{(u+1)} < 2^{(v+1)}$ and $5^{(u+3)} < 5^{(v+3)}$, whereupon

$$(2^{(u+1)}5^{(u+3)})^{(u-2)} < (2^{(v+1)}5^{(v+3)})^{(u-2)} < (2^{(v+1)}5^{(v+3)})^{(v-2)},$$

i.e., $f(u) < f(v)$. Therefore $f(x)$ assumes each of its values exactly once, and so $x = 2$ is the only solution of the given equation. □

Solution 3 The equation has three solutions. Rewriting the equation as

$$(2^{(6-x)}5^{(6-x)})2^{[(x+2)-(6-x)]} = 10^{x^2},$$

we find that

$$2^{2(x-2)} = 10^{(x^2+x-6)} = 10^{(x+3)(x-2)},$$

whence either $x = 2$ or $4 = 2^2 = 10^{(x+3)}$; the latter equation yields $x = 2\log_{10} 2 - 3$.

However, the equation can also be rewritten as $5^{[(6-x)-(x+2)]}10^{(x+2)} = 10^{x^2}$, or

$$5^{2(2-x)} = 10^{(x^2-x-2)} = 10^{(x+1)(x-2)},$$

so that $x = 2$ or $5^{-2} = 10^{(x+1)}$; the latter equation can be simplified to $x = -(1 + 2\log_{10} 5)$.

Therefore, the equation has three solutions: $x = 2$, $x = 2\log_{10} 2 - 3$ and $x = -(1 + 2\log_{10} 5)$. □

This problem appeared on the 2008 *Euclid Contest* for Grade 11 students offered by the Canadian Mathematics Competition of the Centre for Education in Mathematics and Computing at the University of Waterloo in Ontario, Canada. Candidates arrived at the solution in many different ways, often correctly, and there were several forms of the answer. Students might be asked to take logarithms to each of the bases 2, 5 and 10, solve the resulting quadratic equations for x (noting that $x = 2$ is one solution) and then harmonize the forms of the second solution.

Let us review the three solutions. The student who produced Solution 1 was perhaps banking on the unique factorization of an integer as a product of integer powers of primes. However, the uniqueness no longer holds if we allow non-integral exponents. In the second solution, the second inequality fails when $2^{v+1}5^{v+3}$ is less than 1. The third solution consists of two correct solutions that give the same result; note that $2\log_{10} 2-3 = -(1+2\log_{10} 5)$.

2.16 Reciprocating for success

M.A. Khan sends a student's solution to a simultaneous system of equations which uses an unorthodox method of reciprocating to obtain the correct answers.

Problem Solve the following system of equations

$$2u + v = 1; \tag{1}$$

$$\frac{1}{u} + \frac{1}{v} = \frac{3 + \sqrt{3}i}{2}. \tag{2}$$

Solution Taking reciprocals in equation (2) yields

$$u + v = \frac{2}{3 + \sqrt{3}i} = \frac{3 - \sqrt{3}i}{6}. \tag{3}$$

Solving equations (1) and (3) leads to

$$(u, v) = \left(\frac{3 + \sqrt{3}i}{6}, -\frac{1}{\sqrt{3}}i\right).$$

A second solution to the system can be obtained by taking reciprocals in equation (1) to obtain

$$\frac{1}{2u} + \frac{1}{v} = 1. \tag{4}$$

Solving equations (2) and (4) leads to

$$\left(\frac{1}{u}, \frac{1}{v}\right) = \left(1 + \sqrt{3}i, \frac{1 - \sqrt{3}i}{2}\right).$$

so that

$$(u, v) = \left(\frac{1 - \sqrt{3}i}{4}, \frac{1 + \sqrt{3}i}{2}\right). \qquad \square$$

These solutions actually satisfy the given system. Solving (1) and (2) in a more palatable way leads to the quadratic

$$12u^2 - (9 - \sqrt{3}i)u + (3 - \sqrt{3}i) = 0,$$

which is satisfied by both values of u provided in the "solution"; each of these values leads to the corresponding values of v given above.

More generally, the question arises as to when the solution to the system

$$ax + y = p$$

$$\frac{x}{b} + y = \frac{1}{q}$$

is also a solution to the system

$$ax + y = p$$

$$\frac{b}{x} + \frac{1}{y} = q.$$

The given system corresponds to

$$(x, y; a, b, p, q) = \left(u, v; 2, 1, 1, \frac{3+\sqrt{3}i}{2}\right)$$

and

$$(x, y; a, b, p, q) = \left(\frac{1}{u}, \frac{1}{v}; 1, 2, \frac{3+\sqrt{3}i}{2}, 1\right).$$

Solving the first two equations and plugging the solution into the fourth yields the condition

$$(ab-1)^2 = (pq-1)(ab-pq)$$

provided $bq \neq 0$, $ab \neq 1$, $pq \neq 1$ and $ab \neq pq$. The system in the problem satisfies this condition. It is remarkable, if the system was posed by chance, that the student's solution works.

2.17 A lot of values

Problem Solve the diophantine equation

$$x^2 - xyz + (y+z) = 0$$

for integer values of x, y, z.

Solution Upon multiplication by x, we can rewrite the equation as

$$x^3 + 1 = (xy-1)(xz-1),$$

so that

$$(x+1)(x^2-x+1) = (xy-1)(xz-1).$$

We can get some solutions by matching up factors on the two sides of the equation to get the system

$$\begin{aligned} x+1 &= xy-1 \\ x^2-x+1 &= xz-1 \end{aligned}$$

or

$$\begin{aligned} x(y-1) &= 2 \\ z &= x + \frac{2}{x} - 1. \end{aligned}$$

For example, we can satisfy the first with $(x, y) = (-1, -1)$ and use the second to obtain $z = -4$. One can see that the original equation holds with these values. □

This solution is correct as far as it goes. However, the equality of two pairwise products does not imply that corresponding factors are equal. Much more analysis is needed. The cases that $yz = 0$ and $y + z = 0$ are easily disposed of, so we will assume that neither of these conditions holds. The given equation is equivalent to

$$(2x - yz)^2 = y^2z^2 - 4(y + z).$$

Since the right side is a square distinct from y^2z^2, it either is not less than $(|yz| + 1)^2$ or is not greater than $(|yz| - 1)^2$. This implies that $2|yz| \leq |4(y + z) + 1| \leq 4|y| + 4|z| + 1$, which in turn leads to

$$2(|y| - 2)(|z| - 2) \leq 9.$$

From this, we can deduce that the lesser of $|y|$ and $|z|$ does not exceed 4.

Consideration of cases leads to five parametric families of solutions and five sporadic solutions, up to interchanging the values of y and z:

$$(x, y, z) = (0, t, -t), (-t^2, t, -t), (t, 0, -t^2), (-1, -1, t), (t, -1, 1 - t);$$
$$(x, y, z) = (1, 2, 3), (2, 1, 5), (2, 2, 2), (3, 1, 5), (5, 2, 3).$$

2.18 A functional equation

A graduate class was asked to find the solutions to the functional equation

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y,$$

where x and y are reals.

A quick approach is to make two of the arguments of the function equal by making the substitution $y = \frac{1}{2}[x^2 - f(x)]$. Then $f(x) + y = x^2 - y$ and we obtain

$$2f(x)[x^2 - f(x)] = 0.$$

Therefore, there are two solutions $f(x) = 0$ and $f(x) = x^2$. □

However, this does not finish the job. There is a difference between saying that, for each x, either $f(x) = 0$ or $f(x) = x^2$, and making the assertion that either, for all x, $f(x) = 0$ or, for all x, $f(x) = x^2$. All we can conclude

at this point is that there are two subsets of the reals $A = \{x : f(x) = 0\}$ and $B = \{x : f(x) = x^2\}$ whose union is the set of all reals and whose intersection is $\{0\}$. Most of the students failed to pick this up. All we know for sure is that $f(0) = 0$.

Suppose that B contains a nonzero number v. Then, making the substitution, $(x, y) = (v, v^2)$, we find that B also contains $2v^2$ and hence contains infinitely many elements. Let u be a real for which $f(u) = 0$, and suppose that nonzero $v \in B$ is chosen so that $u \neq v^2$. Then

$$f(2v^2 - u) = f(f(v) + (v^2 - u)) = f(u) + 4v^2(v^2 - u) = 4v^2(v^2 - u).$$

Since $f(2v^2 - u) \neq 0$,

$$4v^4 - 4v^2u + u^2 = (2v^2 - u)^2 = f(2v^2 - u) = 4v^2(v^2 - u) = 4v^4 - 4v^2u,$$

whence $u = 0$. Thus, if B contains one nonzero element, it must consist of all the reals. Therefore, there are indeed only two functional solutions, 0 and x^2.

2.19 A surd equation

Problem Solve the equation

$$x + \sqrt{5x + 19} = -1.$$

Solution (by a student)

$$x + \sqrt{5x + 19} = -1 \Longrightarrow x + (\sqrt{5x + 19})^2 = (-1)^2$$
$$\Longrightarrow x + 5x + 19 = 1 \Longrightarrow 6x = -18.$$

Thus the solution is $x = -3$. □

Isolating the surd and squaring gives a quadratic equation with two solutions 6 and -3, the first of which is extraneous. Thanks for this item to Carl Libis.

2.20 Extraneous roots

It is well known that the process of eliminating surds in a surd equation can produce extraneous roots, but here is an extreme example of the phenomenon. The equation to be solved was

$$\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1.$$

One student squared the equation to obtain

$$2x + 11 - 10\sqrt{x-1} + 2\sqrt{(x+3-4\sqrt{x-1})(x+8-6\sqrt{x-1})} = 1$$

which simplifies to

$$\sqrt{x^2 + 35x - 10(x+5)\sqrt{x-1}} = 5\sqrt{x-1} - (x+5).$$

Squaring again and simplifying leads to an identity. So really, after all of this work, we have gained absolutely *no* information about the solution, and everything is open for checking.

Why should this occur? Noting that the equation is equivalent to

$$|\sqrt{x-1} - 2| + |\sqrt{x-1} - 3| = 1,$$

we find that it is satisfied by $5 \leq x \leq 10$, so that it has infinitely many solutions. The strategy of eliminating the surds by squaring leads to a polynomial equation that has at least this many roots, and so must be an identity.

2.21 Right on target!

Larry Braden sends this student solution:

Exercise. Solve the equation

$$\sqrt{3x-2} - \sqrt{2x-3} = 1.$$

Solution. Squaring both sides, we obtain

$$\begin{aligned} 3x - 2 - 2x - 3 = 1^2 &= 1 \\ x - 5 &= 1 \\ x &= 6. \square \end{aligned}$$

The answer is, in fact, correct!

As with all such howlers, it is interesting to ask whether there is a more general set of circumstances under which the process gives the right answer. Suppose that the surd equation to be solved is

$$\sqrt{ax-b} - \sqrt{cx-d} = 1$$

and that the student's reduction leads to

$$ax - b - cx - d = 1.$$

Basically, we want to find when two equations for x can be solved simultaneously, one being the linear equation of the student's solution. The least informative answer is to solve the linear equation and plug the solution into the surd equation, and state the result as a condition on the parameters for the two equations to have a common solution. But if we ask students for a conveniently described parameterized set of values for (a, b, c, d) for which the equations are satisfied and then to check that it works, then we are forcing them to find some strategy to cut through the algebraic thicket.

$$\begin{aligned}\sqrt{ax-b}-\sqrt{cx-d}=1 &\Longrightarrow ax-b=1+cx-d+2\sqrt{cx-d}\\ &\Longrightarrow (a-c)x=(b+1-d)+2\sqrt{cx-d}.\end{aligned}$$

To make the manipulation simpler, specialize to $a = c + 1$ and $d = b + 1$. Then the equation becomes

$$x^2 = 4(cx - d) \Longrightarrow 0 = x^2 - 4cx + 4d.$$

Using the student's "method" to solve the same equation gives $ax - b - cx - d = 1$ which yields $x = (1 + b + d)/(a - c) = 2d$. So, for the "method" to work, we need

$$0 = 4d^2 - 8cd + 4d = 4d(d - 2c + 1)$$

which can be achieved by making $2c = d + 1$. So we can take

$$(a, b, c, d) = (t + 1, 2(t - 1), t, 2t - 1)$$

for some real t. The original problem corresponds to $t = 2$.

The equation

$$\sqrt{(t+1)x-2(t-1)}-\sqrt{tx-(2t-1)}=1$$

is satisfied by $x = 2$ and $x = 4t - 2$. The first solution works for all values of t, while the second is valid if and only if $t \geq \frac{1}{2}$. The equation $(t + 1)x - 2(t - 1) - tx - (2t - 1) = 1$ is equivalent to $x = 4t - 2$.

Approach 2. Begin with some analysis. We want to solve simultaneously the equations

$$\sqrt{ax-b}-\sqrt{cx-d}=1 \tag{1}$$

and

$$ax - b - cx - d = 1. \tag{2}$$

From (1), we find that

$$ax - b = 1 + (cx - d) + 2\sqrt{cx - d}. \tag{3}$$

From (2) and (3), we obtain that $d = \sqrt{cx - d}$, so that $x = (d^2 + d)/c$. From (2), we have that $x = (1 + b + d)/(a - c)$.

Select a, c, d so that $d > 0$ and $ac(a - c) \neq 0$, and choose b to satisfy

$$\frac{d^2 + d}{c} = \frac{1 + b + d}{a - c}.$$

Let

$$x = \frac{d^2 + d}{c} = \frac{1 + b + d}{a - c} = \frac{(d^2 + d) + (1 + b + d)}{c + (a - c)} = \frac{(d + 1)^2 + b}{a}.$$

Then

$$\sqrt{ax - b} - \sqrt{cx - d} = \sqrt{(d + 1)^2} - \sqrt{d^2} = (d + 1) - d = 1$$

and

$$ax - b - cx - d = (d + 1)^2 + b - b - d^2 - d - d = 1.$$

If we take, in particular, $c = d = 1$, then we are led to the family of parameters $(a, b, c, d) = (a, 2a - 4, 1, 1)$.

Finally, simply eyeballing the situation and trying things out might lead to $(a, b, c, d) = (2c, 0, c, 1)$.

2.22 An "artifice" of Hall and Knight

The author of this item is John Webb of the University of Cape Town, South Africa.

Questions involving equations in which square roots of polynomials appear may often be resolved by judicious squaring of the equation until all square roots have disappeared. Such a method has two disadvantages. The squaring process is likely to produce extra solutions, and the resulting root-free polynomial may have an inconveniently high degree.

In Hall and Knight's *Higher Algebra* (4th edition, reprinted in 1957) we find the following example (article 132, page 99) in which an alternative procedure, by using "conjugate surds," solves the given equation rather neatly.

132. The following artifice is sometimes useful.

Example. Solve

$$\sqrt{3x^2 - 4x + 34} + \sqrt{3x^2 - 4x - 11} = 9. \tag{1}$$

We have *identically*

$$(3x^2 - 4x + 34) - (3x^2 - 4x - 11) = 45. \tag{2}$$

Divide each member of (2) by the corresponding member of (1); thus

$$\sqrt{3x^2 - 4x + 34} - \sqrt{3x^2 - 4x - 11} = 5. \tag{3}$$

Now (2) is an *identical equation* true for *all* values of x, whereas (1) is an equation which is true only for certain values of x; hence also equation (3) is only true for these values of x.

From (1) and (3) by addition

$$\sqrt{3x^2 - 4x + 34} = 7;$$

whence $x = 3$, or $-\frac{5}{3}$.

The two solutions do indeed satisfy the original equation.

However, not all is well with Hall and Knight's "artifice."

Suppose we start with a very similar equation, obtained simply by changing the "9" on the right-hand side to a "5."

Then, using the same identity as before, and dividing out, we obtain the same conjugate surd, but with a "9" on the right-hand side. After adding the two surd expressions, we obtain exactly the same equation (3) as before, with, of course, the same solutions. But now neither of the two solutions satisfies the original equation!

132′. The following artifice is sometimes useful.

Example. Solve

$$\sqrt{3x^2 - 4x + 34} + \sqrt{3x^2 - 4x - 11} = 5. \tag{1}$$

We have *identically*

$$(3x^2 - 4x + 34) - (3x^2 - 4x - 11) = 45. \tag{2}$$

Divide each member of (2) by the corresponding member of (1); thus

$$\sqrt{3x^2 - 4x + 34} - \sqrt{3x^2 - 4x - 11} = 9. \tag{3}$$

Now (2) is an *identical equation* true for *all* values of x, whereas (1) is an equation which is true only for certain values of x; hence also equation (3) is only true for these values of x.

From (1) and (2) by addition

$$\sqrt{3x^2 - 4x + 34} = 7;$$

whence $x = 3$, or $-\frac{5}{3}$.

Why does Hall and Knight's "artifice" work in the first case but not in the second?

The answer is by no means obvious, and the false step is very cunningly hidden. Indeed, the incorrect step is a feature of the standard methods of solving equations at school or college level that is often glossed over. The point is neatly made in J.E. Littlewood's *Mathematical Miscellany* (page 41):

Schoolmaster: 'Suppose x is the number of sheep in the problem.'

Pupil: 'But, Sir, suppose x is not the number of sheep.'

I asked Prof. Wittgenstein was this not a profound philosophical joke, and he said it was.

The point is simply that, if we assume that an equation has a solution when it does not, we are introducing a false assumption into our discussion, and we should not be surprised by the resulting *reductio ad absurdum*.

Hall and Knight's original equation does indeed have a solution, but the second does not. An astute reader might have noticed this from working out the minimum value of the quadratic expression under the first square root, and observing that it was greater than 25. Since the second surd term is nonnegative, the second equation has no solution.

Now the question is: at what step in the solution of the second equation were the two false solutions introduced? Extraneous solutions may arise when an irreversible step is used, such as squaring out.

Looking again at Hall and Knight's solution procedure, we note that the step of adding the two equations (1) and (3) to obtain just one equation cannot be reversed. That is where the false solutions were introduced.

Comment. A simpler equation that leads to the same issue is

$$\sqrt{x+3} + \sqrt{x-2} = 5.$$

More generally, if we are given the equation

$$\sqrt{x+a} + \sqrt{x+b} = u$$

to solve, then we can use the fact that the product of the left side and its surd conjugate is equal to $a - b$ to get

$$\sqrt{x+a} - \sqrt{x+b} = v$$

where $uv = a - b$. We find that $\sqrt{x+a} = \frac{1}{2}(u+v)$ and $\sqrt{x+b} = \frac{1}{2}(u-v)$. We can square either one of these to find that x satisfies

$$4x = (u+v)^2 - 4a = (u-v)^2 - 4b$$

with $v = (a-b)/u$. If $u+v$ and $u-v$ both turn out to be positive, then we have a solution to the equation. For example, $\sqrt{x+3} + \sqrt{x-2} = 5$ is satisfied by $x = 6$.

However, I was struck by the Hall and Knight passage as an example of the opacity that seemed to afflict many of the British texts of the early twentieth century. With such a nice derivation of (3) using the relationship

$$(\sqrt{f(x)} + \sqrt{g(x)})(\sqrt{f(x)} - \sqrt{g(x)}) = f(x) - g(x)$$

available, they chose rather to pull the rabbit (2) out of a hat. Their derivation of (3) is quite pernicious.

2.23 An appeal to symmetry

Problem. Suppose that $a, b, u, v \geq 0$ and that $a^5 + b^5 \leq 1$, $u^5 + v^5 \leq 1$. Prove that

$$a^2u^3 + b^2v^3 \leq 1.$$

Solution (from a student). Note that

$$\begin{aligned}(a^5+b^5)(u^5+v^5) - (a^2u^3+b^2v^3)(a^3u^2+b^3v^2) &= a^5v^5 + b^5u^5 - a^2b^3u^3v^2 - a^3b^2u^2v^3 \\ &= (a^3v^3 - b^3u^3)(a^2v^2 - b^2u^2) \\ &= (a^2v^2 + abuv + b^2u^2)(av - bu)^2(av + bu) \\ &\geq 0.\end{aligned}$$

Hence

$$(a^2u^3 + b^2v^3)(a^3u^2 + b^3v^2) \leq 1.$$

Since the product of the two factors on the left does not exceed 1, at least one of the factors must not exceed 1. Suppose that it is the second factor, *i.e.*, $a^3u^2 + b^3v^2 \leq 1$. Now interchange the roles of (a, b) and (u, v) to obtain $u^3a^2 + v^3b^2 \leq 1$. The result now follows. □

Comment. Making the interchange yields the result that

$$(u^2a^3 + v^2b^3)(u^3a^2 + v^3b^2) \leq 1,$$

which is possible with the first factor on the right less than 1 and the second greater.

2.24 An inequality

Proposition *Let $x, y > 0$. Then*

$$\frac{x}{(x+y)} \leq \frac{1}{(1+x)}.$$

Proof We have that

$$\frac{x}{(1+x)(x+y)} = \frac{1}{(1+x)(1+(y/x))} \leq \frac{1}{(1+\sqrt{y})^2}$$

by the Cauchy-Schwarz Inequality $(a_1^2 + a_2^2)(b_1^2 + b_2^2) \geq (a_1b_1 + a_2b_2)^2$ applied to $a_1 = b_1 = 1$, $a_2 = \sqrt{x}$ and $b_2 = \sqrt{y/x}$. Equality occurs when $x = y/x$, or $y = x^2$. Hence, maximizing, we see that

$$\frac{x}{(1+x)(x+y)} \leq \frac{1}{(1+x)^2}. \qquad (\square)$$

Comment In the event that $y < x^2$, then $(1+x)^{-2} < (1+\sqrt{y})^{-2}$ and the second inequality in the "proof" does not follow from the first. Check out the case $(x, y) = (2, 3)$.

2.25 An Asian-Pacific inequality

On the 2004 Asian-Pacific Mathematics Olympiad, the last question was a tough inequality. The solution was provided after the examination by a student.

Problem Prove that

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca)$$

for $a, b, c > 0$.

Solution By an application of the arithmetic-geometric means inequality, we have that

$$\begin{aligned}(a^2+2)(b^2+2)(c^2+2)&\\ &= a^2b^2c^2 + 2(a^2b^2+b^2c^2+c^2a^2) + 4(a^2+b^2+c^2) + 8\\ &\geq a^2b^2c^2 + 6a^{4/3}b^{4/3}c^{4/3} + 12a^{2/3}b^{2/3}c^{2/3} + 8\\ &= [(abc)^{2/3}+2]^3,\end{aligned}$$

with equality if and only if $a = b = c$. On the other hand, it is straightforward to establish that

$$9(ab+bc+ca) \leq 9(a^2+b^2+c^2),$$

again with equality if and only if $a = b = c$.

Thus, it suffices to establish that the inequality

$$9(a^2+b^2+c^2) \leq [(abc)^{\frac{2}{3}}+2]^3$$

holds when $a = b = c = k$, say. But this follows from

$$\begin{aligned}(k^2+2)^3 - 27k^2 &= k^6 + 6k^4 - 15k^2 + 8\\ &= (k^2-1)^2(k^2+8) \geq 0.\end{aligned}$$

Equality occurs if and only if $a = b = c = k = 1$. □

Comment However, note that, when $(a, b, c) = (8, 1, 1)$,

$$594 = (a^2+2)(b^2+2)(c^2+2) > [(abc)^{2/3}+2]^3 = 216$$

while

$$153 = 9(ab+bc+ca) < 9(a^2+b^2+c^2) = 594.$$

2.26 Maximizing a rational function

Problem Determine the maximum value of

$$\frac{xy + 2yz + zw}{x^2+y^2+z^2+w^2}$$

over all quadruples (x, y, z, w) of real numbers not all zero.

Solution 1 By the arithmetic-geometric means inequality, $2xy \le x^2 + y^2$, $2yz \le y^2 + z^2$, $2zw \le z^2 + w^2$, from which we see that the given expression does not exceed

$$\frac{(x^2 + y^2 + z^2 + w^2) + 2(y^2 + z^2)}{2(x^2 + y^2 + z^2 + w^2)} = \frac{1}{2} + \frac{y^2 + z^2}{x^2 + y^2 + z^2 + w^2}. \quad (*)$$

Equality holds if and only if $x = y = z = w$, and so the maximum value is equal to 1, and occurs when all the variables are equal. □

Solution 2 As in Solution 1, we can obtain the inequality $(*)$. The right side of the inequality is maximized when $x = w = 0$, and so the maximum value of the expression given in the problem is $\frac{1}{2} + 1 = \frac{3}{2}$. □

Solution 3 For all positive values of a, b, c, we have that $a^2x^2 + y^2 \ge 2axy$, $b^2y^2 + z^2 \ge 2byz$ and $c^2z^2 + w^2 \ge 2czw$, from which it follows that

$$\frac{xy + 2yz + zw}{x^2 + y^2 + z^2 + w^2} \le \frac{(a/2)x^2 + ((1/2a) + b)y^2 + ((1/b) + (c/2))z^2 + (1/2c)w^2}{x^2 + y^2 + z^2 + w^2}.$$

To find the maximum, set

$$\frac{a}{2} = \frac{1}{2a} + b = \frac{1}{b} + \frac{c}{2} = \frac{1}{2c}.$$

Then $b = (a^2 - 1)/2a$, $c = 1/a$ and

$$\frac{a}{2} = \frac{2a}{a^2 - 1} + \frac{1}{2a} \qquad \text{or} \qquad \frac{a^2 - 1}{2a} = \frac{2a}{a^2 - 1}.$$

Therefore $a^2 - 1 = 2a$. Substituting in the root $a = 1 + \sqrt{2}$, we find that the right side of the foregoing inequality is $\frac{1}{2}(1 + \sqrt{2})$, and this is the desired maximum. □

Solution 4 We first make the observation that, if p, q, r, s are reals with q and s positive, then $(p + r)/(q + s)$ lies between p/q and r/s, with equality if and only if $ps = rq$. Making the substitution $x = a + b$, $y = c + d$, $z = c - d$ and $w = a - b$, the expression of the problem becomes

$$u \equiv \frac{ac + bd + c^2 - d^2}{a^2 + b^2 + c^2 + d^2}.$$

Taking $(p, q, r, s) = (bd - d^2, b^2 + d^2, ac + c^2, a^2 + c^2)$, and noting, since the values of a, b, c, d are arbitrary, that $bd - d^2$ is clearly less than $ac + c^2$, we have that

$$\frac{bd - d^2}{b^2 + d^2} \leq u \leq \frac{ac + c^2}{a^2 + c^2}.$$

Since the left and right members of this inequality are independent, it suffices to maximize the right side. It turns out that the maximum value of the right side is $\frac{1}{2}(\sqrt{2} + 1)$, attained when $(a, c) = (\sqrt{2} - 1, 1)$, and this is the desired maximum of the given expression. □

Comments These are based on solutions offered by students. The first two are not hard to troubleshoot. In the case of the third, the ingredients are present, but it is not clear why one should equate the coefficients on the right side of the displayed inequality. First, one should note that it suffices to maximize the rational function when x, y, z, w are nonnegative, not all zero. The opening three inequalities are valid, with equality when $ax = y$, $by = z$ and $cz = w$. Now select

$$(a, b, c) = (1 + \sqrt{2}, 1, -1 + \sqrt{2});$$

then

$$\frac{a}{2} = \frac{1}{2a} + b = \frac{1}{b} + \frac{c}{2} = \frac{1}{2c}.$$

Then the rational function does not exceed $\frac{1}{2}(1 + \sqrt{2})$, equality occurring when $x : y : z : w = 1 : (1 + \sqrt{2}) : (1 + \sqrt{2}) : 1$.

The difficulty with the fourth solution is somewhat subtle. First, it is not clear that *every* value of $f \equiv (bd - d^2)/(b^2 + d^2)$ is less than *every* value of $g \equiv (ac + c^2)/(a^2 + c^2)$, but this is easily patched, as f is less than $(bd + d^2)/(b^2 + d^2)$ which does not exceed the maximum value of g. However, even if we maximize g, we can apparently ensure that $u = g$ only if we can select b and d to make $f = g$. But there is another way out, $u = g$ when $b = d = 0$, and this is precisely what occurs in the optimizing situation.

2.27 A plausible inequality

Problem Let $n \geq 1$. Is it true that, for any $2n + 1$ positive real numbers $x_1, x_2, \ldots, x_{2n+1}$, we have that

$$\frac{x_1 x_2}{x_3} + \frac{x_2 x_3}{x_4} + \cdots + \frac{x_{2n+1} x_1}{x_2} \geq x_1 + x_2 + \cdots + x_{2n+1},$$

with equality if and only if all the x_i are equal?

Solution (by a student) The inequality holds. By reversing the order of the variables, we see that it is equivalent to showing that

$$\begin{aligned}\frac{x_{2n+1}x_{2n}}{x_{2n-1}} + \frac{x_{2n}x_{2n-1}}{x_{2n-2}} + \cdots + \frac{x_3x_2}{x_1} + \frac{x_2x_1}{x_{2n+1}} + \frac{x_1x_{2n+1}}{x_{2n}} \\ \geq x_{2n+1} + x_{2n} + \cdots + x_1 \\ = x_1 + x_2 + \cdots + x_{2n+1}.\end{aligned}$$

Let $x_{2n+2} = x_1$ and $x_{2n+3} = x_2$. For $1 \leq k \leq 2n+1$, we can pair each term x_kx_{k+1}/x_{k+2} in the first inequality with the term $x_{k+2}x_{k+1}/x_k$ in the second, and use the arithmetic-geometric means inequality to deduce that

$$\frac{x_kx_{k+1}}{x_{k+2}} + \frac{x_{k+2}x_{k+1}}{x_k} \geq 2x_{k+1}$$

with equality if and only if $x_k = x_{k+2}$. Hence

$$\begin{aligned}\left(\frac{x_1x_2}{x_3} + \frac{x_2x_3}{x_4} + \cdots + \frac{x_{2n+1}x_1}{x_2}\right) \\ + \left(\frac{x_{2n+1}x_{2n}}{x_{2n-1}} + \cdots + \frac{x_3x_2}{x_1} + \frac{x_2x_1}{x_{2n+1}} + \frac{x_1x_{2n+1}}{x_{2n}}\right) \\ \geq 2(x_1 + x_2 + \cdots + x_{2n+1}),\end{aligned}$$

and the result follows. □

This argument leads to success in the case $n = 1$, when the left side of the inequality remains unchanged when we interchange x_1 and x_3. However, the left side generally does change when $n \geq 2$. Consider this example for $n = 2$: $(x_1, x_2, x_3, x_4, x_5) = (81, 3, 9, 1, 1)$. All we can conclude is that either the left side or its variant obtained by reversing the order of the variables must exceed the right side of the inequality.

2.28 The kinematics of jogging

Ralph Boas (1912–1992), a former professor at Northwestern University, contributed a guest essay to the text, *Calculus* by Monty J. Strauss, Gerald L. Bradley, and Karl J. Smith (3rd ed., Prentice Hall, 2002, pp. 351–352). In it, he raised an interesting question about jogging. Suppose that you average 8 minutes per mile; must you cover some one continuous mile in exactly 8 minutes? Boas pointed out that the answer to this question depends on whether the total distance is an integer number of miles. More precisely,

if you run an integer number of miles, then some mile must be covered in exactly 8 minutes. However, if you run a non-integer number of miles, then it may happen that no continuous mile is run in exactly 8 minutes.

Hanxiang Chen agrees with these conclusions and accepts Boas' argument for a run of an integer number of miles. However, he does not agree with Boas' counterexample for the second part of the result.

Boas supposes that you jog so that the time to cover the first x miles of an $n-$mile journey is

$$J(x) = k \sin^2 \frac{\pi x}{n} + 8x$$

minutes, where n is not an integer and $k < 8n/\pi$ is small enough that $J(x)$ is an increasing function. He asserted that $J(x+1) - J(x)$ cannot be 8. However, Chen observes that, when $n > 1$, then $0 < \frac{n-1}{2} < \frac{n+1}{2} < n$ and

$$\begin{aligned} J\left(\frac{n}{2}+\frac{1}{2}\right) - J\left(\frac{n}{2}-\frac{1}{2}\right) &= k\left[\sin^2\left(\frac{\pi}{2}+\frac{\pi}{2n}\right) - \sin^2\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)\right] + 8 \\ &= k\left(\cos^2\frac{\pi}{2n} - \cos^2\frac{\pi}{2n}\right) + 8 = 8, \end{aligned}$$

so that you do cover the interval $[(n-1)/2, (n+1)/2]$ in exactly 8 minutes.

Chen suggests a counterexample that works. Suppose, again that n is a noninteger and that $0 < k < 1/\pi$, and define

$$J(x) = 8n\left(\frac{x + k\sin^2 \pi x}{n + k \sin^2 \pi n}\right)$$

for $0 \le x \le n$. Then $J(x)$ is increasing and $J(n)/n = 8$, so that you average 8 minutes per mile. On the other hand, for $x \in [0, n-1]$, we see that

$$\begin{aligned} J(x+1) - J(x) &= 8n\left(\frac{1 + k(\sin^2 \pi(x+1) - \sin^2 \pi x)}{n + k \sin^2 \pi n}\right) \\ &= 8\left(\frac{n}{n + k \sin^2 \pi n}\right) < 8, \end{aligned}$$

since n nonintegral implies that $\sin^2 \pi n > 0$. Thus, you will never cover one continuous mile in 8 minutes.

2.29 Solar power

Ed Dubinsky, who lived in New York State, presented a problem in a recent internet discussion. I will let him describe the situation and leave it for the reader to adjudicate the alternatives:

I have installed a set of solar panels next to my house which are powerful enough to provide a significant portion of my energy needs. I am still connected to the grid, so on days when there is not enough sun, the energy comes from *NiMo* (our local derivative for the Niagara Mohawk Electric Company) and I pay the standard rate per kilowatt-hour (kwh). On the other hand, if it is really sunny and my panels produce more than I need, then the excess is sent back to the grid and *NiMo* subtracts the number of kwh from what they sent me before determining how much to bill me. It should be noted that the determination of the kwhs is continuous (to the extent that a physical process can be continuous) and so in any given period there are three numbers of kwhs: what my panels produce, what is sent back to *NiMo* and what I get from *NiMo*.

So here is the problem. In a given period, my solar panels produce A kwh. Of that C kwh is not needed for my house and so it is sent back to *NiMo*. During the same period, I receive B kwhs from *NiMo*. During this period, what percentage of my power needs are supplied by my solar panels?

There are two candidates for the answer:

$$\frac{100A}{A+B-C} \tag{1}$$

and

$$\frac{100(A-C)}{A+B-2C}. \tag{2}$$

An argument for (1) is that the panels produce A kwh of one's total consumption of $A+B-C$. As for the second, observe that, of the total consumption, the panels provide $A-C$ kwh while the grid provides $B-C$. Thus, the fraction provided by the panels is $(A-C)/[(A-C)+(B-C)]$.

This is a nice problem for classroom discussion, and Ed Dubinsky suggests some other questions for school students to consider:

(a) Does one formula always give a greater percentage?

(b) Suppose the period is one week. Suppose you then collect the data and make the calculation for the following week. Is the percentage for the two week period the average of the percentage for the individual weeks? Why, or why not?

(c) So far, we have assumed that $B > C$. What happens in the event that $B < C$?

2.30 Logarithmic behavior as metaphor

Norton Starr, Amherst College, Amherst, MA, contributes the following essay:

> The public confusion of logarithmic growth with exponential growth first drew my attention at the 1966 Opening Convocation lecture to our college. In the course of this talk, the speaker said, "As opposed to all other appetites which are stimulated by deprivation and satisfied by food, good education stimulates with plenty so that appetite for knowledge and understanding escalate logarithmically to insatiability." Afterwards, I gently tried to explain that exponential growth would more appropriately suggest the intended rapidity. Subsequently, I have come across a possible explanation for the use of "logarithmically" to describe rapid increase: In bivariate data representation, the adjustment of a variable so as to obtain a straight line relation is often desirable, as Mosteller, Rourke and Thomas pointed out in their classic text [6]. Many data are "straightened out" by plotting the logarithm of one variable against the other variable. In particular, exponential growth when plotted on such "semi-log" paper shows up as a linear graph. (The increments on the vertical scale of semi-log paper are unequal in value; they are an exponential of the equally spaced values on a traditional $y-$axis.) In some disciplines, such as biology, "logarithmic growth" actually refers to what we mathematicians know as exponential growth because of this method of linearizing the data. That must underlie one of the most prominent usages, nuclear scientist Ralph Lapp's *The Logarithmic Century*, in which he writes about the rapid growth of resource consumption, *etc.* [4]. Interestingly, the dust jacket blurb uses, instead, the terminology "exponential growth." I would expect Lapp to have been aware of a choice of descriptions, and why he opted for "logarithmic" is beyond me. In any case, he helped keep alive an unfortunate terminology.
>
> Since the occasion of the Opening Convocation, I have run across other such examples, largely from sources with no apparent scientific culture to have misled them. Some of these are recounted below. Although people still misuse the terminology, many display an awareness of the proper distinctions at issue. Perhaps with the help of those who teach about rates of change, computer algorithms, L'Hôpital's Rule, and the like, "logarithmic growth" and "exponential growth" will avoid the fate of "reticent" and "reluctant," which have become essentially synonymous.

Here are three examples of metaphorical growth from the *New York Times*, with the third suggesting an improved understanding on the part of this newspaper:

- from a review of Harlow's Shapley's autobiography: "if the autobiographer opts for a method he believes will grant him immortality without industry, his risks rise logarithmically." [8]
- from a story about corruption and drugs: " 'The drug situation is a horror story, increasing logarithmically.'" [9]
- from a more recent story: "Street crime, fed by an explosion of drug abuse, has risen exponentially." [10]

These sorts of examples also come up on television, as shown in this quote from Tim Daggett's *NBC* Olympics gymnastic commentary, July 23, 1996: "We have seen Kerri Strug's confidence increase almost logarithmically this year." [5]

Not even my own colleagues are immune to the confusion, as the following quote from our college's alumni magazine captures. A member of the English department, with a particular interest in linguistics, complimented a student: "I observed that the experience of doing a translation made her grow logarithmically." [1]

A fellow mathematician brought to my attention the following gaffe, by a nationally prominent pediatrician: "The stresses parents are under have increased logarithmically since I was raising my children in the 60s and 70s." [3]

Of course, the Internet, a source of so much of value yet also so much error and disinformation, offers a host of examples, especially self-referential ones. For instance:

- from a report on the state of technology: "The Internet is growing logarithmically." [13]
- from Hunter S. Thompson on *ESPN*: "the Meanness Quotient of the US image in the world is growing logarithmically with every passing day." [14]
- from an editor at O'Reilly & Associates, which specializes in books on *Linux* and programming: "the logarithmic growth of the Web." [15]
- and from a sales manager for *Sun Microsystems*, "The Internet's Logarithmic Growth. Websites Doubling Every 57 Days." [16]

As John Allen Paulos has noted, the confusion also arises in nontechnical commercial promotions: "One real estate development

company advertised that an investment with it would grow logarithmically." [11]

Finally, as a result of my discussing this topic when teaching calculus, a student, Bill Nahill, spotted the following juicy example: "Following his death, Malcolm's influence expanded in dramatic, almost logarithmic, fashion." [12] Malcolm X was a courageous fellow, and deserves better than this.

The above quotes evidence a widespread public confusion, extending even to those who should know better. However, it is worth giving two examples of writers in the arts who make proper use of logarithm as metaphor. Rather than suggesting a rapidity of growth, each uses the logarithm to connote sheer complexity. In his 1947 review of a film noir classic, Bosley Crowther wrote, "the sum of deceitful complications that occur in 'Out of the Past' must be reckoned by logarithmic tables, so numerous and involved do they become." [7] More cryptically, the celebrated poet John Ashbery's most famous work, "Self-Portrait in a Convex Mirror" contains the description:

"New York
Where I am now, which is a logarithm
Of other cities." [2]

It looks as though these two prominent writers have perfect pitch in this area! Unfortunately, commentators across a broad range of careers still mangle the subject. With luck, the examples presented here will encourage *CMJ* readers to do their part to stamp out this form of illiteracy.

References

[1] *Amherst* (Summer, 1999), 19.

[2] John Ashbery, *Self-Portrait in a Convex Mirror*, Viking Penguin, 1985, p. 195.

[3] Interview with T. Berry Brazelton, *Boston Globe Magazine*, October 29, 2000, p. 10.

[4] Ralph E. Lapp, *The Logarithmic Century: Charting Future Shock*, Prentice-Hall, 1973, p. 21.

[5] Miscellaneous quotation [25] from the Mathematical Olympiad Summer Program quotations list of Michael Freiman, `mail.med.upenn.edu/~freiman/MOPQuote.htm`.

[6] Frederick Mosteller, Robert E.K. Rourke, and George B. Thomas, Jr., *Probability with Statistical Applications*, Second edition, Addison-Wesley, 1970, p. 386.

[7] *The New York Times*, November 26, 1947, p. 18.

[8] Earl Ubell, Review of the book *Through Rugged Ways to the Stars* by Harlow Shapley (Scribners, 1969), *The New York Times Sunday Book Review*, July 13, 1969, p. 7.

[9] Remark by Customs Commissioner von Raab, *The New York Times*. May 12, 1986, p. 1.

[10] Report by Tim Golden, *The New York Times*, January 9, 2000, p. 1.

[11] John Allen Paulos, *A Mathematician Reads the Newspaper*, Harper-Collins, 1995, p. 88.

[12] William L. Van Deburg, *New Day in Babylon: The Black Power Movement and American Culture, 1965–1975*, University of Chicago Press, 1992, p. 2.

[13] `www.ndu.edu/sdcfp/FedEx%20Final%20Report.doci` (June, 2000, report on Federal Express for the Secretary of Defense Corporate Fellows Program, p. 24.)

`www.hsmai.org/releaseDetail.cfm?id=130`

The source cited is described as "an expert on technology" at a speakers' bureau specializing in academia: `www.lordly.com/lordly/` (under Future Trends & Technology).

[14] `http://espn.go.com/page2/s/thompson/020328.html` (11th paragraph)

[15] `www.empowermentzone.com/cgiintro.txt`

[16] `www.ifla.org/IV/ifla62/62-kenc.htm` (Last lines of Chapter II, "Sun and the Internet," just before "Java.")

Addendum. As I prepared for this book, Norton Starr told me that this usage of "logarithmic" has not abated. He referred to an article by Brian Gilmore, *A Conservative for Impeachment*, which appeared in *The Progressive* for December, 2006, on pages 23–24 (and `www.progressive.org/node/4252`) in which he quotes Bruce Fein to the effect that George Bush's signing statements " have multiplied logarithmically."

In the following year, an article by Andrew Sullivan entitled *Goodbye to All That* in *The Atlantic* 300:5 (December, 2007), 40–54, notes that, under President Obama, "America's soft power has been ratcheted up not a notch, but a logarithm."

Note

Related items involving inequalities can found in Chapter 3, Items 4 and 8. A functional equation occurs in Chapter 5, Item 3.

CHAPTER 3

Geometry

3.1 Proof that a 3-4-5 right triangle does not exist

With inadequate care in the use of the double-angle formula, we can conclude that the standard 3-4-5 right triangle does not exist. These arguments are due to Jeff Suzuki.

Argument 1. If θ is the angle opposite the side of length 4, we have $\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta}$. Plugging this formula into the calculator with $\sin\theta = 0.8$ we find that $\sin 2\theta \approx 0.96$, $\sin 4\theta \approx 0.5376$, $\sin 8\theta \approx 0.9066$, and so forth, so that all of θ, 2θ, 4θ, and so on, must be between $0°$ and $180°$. We conclude that $\theta = 0$. □

Argument 2. Start with $\cos\theta = 3/5$ and use $\cos 2\theta = 2\cos^2\theta - 1$. Then $\cos\theta = 0.6$, $\cos 2\theta = -0.28$, $\cos 4\theta \approx -0.8432$ and $\cos 8\theta \approx 0.4219$. Since the cosines of 2θ and 4θ are negative, then 2θ and 4θ must be second or third quadrant angles; hence $90° < 2\theta, 4\theta < 270°$. In particular, θ must exceed $45°$. Since $\cos 8\theta$ is positive, it is a first or fourth quadrant angle, so that $0° < 8\theta < 90°$ or $270° < 8\theta < 360°$. Thus, θ must be less than $45°$, contradicting what we said earlier. □

The difficulty in the first argument is the neglect of the sign of the square root of the function $1 - \sin^2$, and in the second the failure to recognize that the multiple angles could exceed $360°$. However, by noting the signs of the cosines of the angles $2^k\theta$, and placing these in the right ranges, it is possible to approximate more and more closely what θ must be.

3.2 Important knowledge about triangles

In July 2004, an Associated Press item recounted how Governor Jeb Bush of Florida was stumped by a mathematics question from the *Florida Compre-*

hensive Assessment Test, when he addressed a secondary school audience.

> A student, Luana Marques, 18, posed the question, "What are the angles on a three-four-five triangle?" The governor gave a steely grin. "The angles would be — if I was going to guess," he said. "Three-four-five. Three-four-five. I don't know. 125, 90 and whatever remains on 180?" Ms. Marques had the correct answer: 30, 60 and 90.
>
> "The fact that a 51-year old man can't answer a question is not relevant," Mr. Bush, a Republican, said. "You're still going to have to take the *FCAT* and you're still going to have to pass it to get a high school degree."

There are some interesting attributes that would make this a useful item for classroom discussion. I wonder, for example, if the governor's political affiliation would have been mentioned were he a Democrat. And whether the editorial oversight on mathematical correctness might have been more acute.

3.3 The perimeter of a triangle

Peiyi Zhao came across the following problem for students in social sciences in China; it is likely that the examiner had no intention of laying a trap. He showed this question to colleagues, and what follows below is perhaps the intended solution. He remarks on how interesting it is to see how various people approached this question and on how very few noticed the flaw.

Problem In a given triangle ΔABC, the ratio of the angles is given by $A : B : C = 2 : 3 : 4$ and that of two of the sides by $AB : BC = 3 : 2$. Also, $AC = 5$. Find the perimeter of ΔABC.

Solution The angles are 40°, 60° and 80°. Assume that the respective lengths of AB and BC are $3y$ and $2y$. Then

$$25 = 9y^2 + 4y^2 - 12y^2 \cos 60^\circ = 7y^2$$

whence $y = (5\sqrt{7})/7$. The perimeter is found to be

$$\frac{25}{7}\sqrt{7} + 5. \qquad \square$$

The flaw is that too much information is given, so that the data are inconsistent. The ratio of two sides is equal to the ratio of the sines of the

angles respectively opposite, so that $3 : 2$ should be equal to the ratio $\sin 80° : \sin 40° = 2\cos 40° : 1$. One way to see that $\cos 40°$ is not equal to $3/4$ is to check out the formula $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ with $\theta = 40°$.

3.4 Conditions of equality

The following problem exemplifies an error of a type that I have seen frequently in student solutions. It is seductive because the answer obtained is often correct.

Problem. Given that A, B and C are the angles of a triangle, determine the maximum value of $\sin A \sin B \sin C$.

Solution. By the arithmetic-geometric means inequality,

$$\sin A \sin B \sin C \leq \left(\frac{\sin A + \sin B + \sin C}{3}\right)^3.$$

Equality occurs if and only if $\sin A = \sin B = \sin C$, which occurs exactly when $A = B = C = 60°$. Thus, the desired maximum occurs when all the angles are equal and is $3\sqrt{3}/8$. □

However, it is conceivable that both sides take larger values for other triples.

3.5 When isosceles gives maximum area

Problem. Let A and B be fixed points in the plane. Find all positive reals k for which the following assertion holds:

among all triangles ABC with $|AC| = k|BC|$, the one with the largest area is isosceles.

Solution (by a student). When $k = 1$, the locus of C is the right bisector of AB, every triangle is isosceles and there is no one with the largest area. When $k \neq 1$, the case $|AC| = |BC|$ can never occur. In this case, locate the points A and B at $(0, 0)$ and $(1, 0)$ respectively in the cartesian plane. Let C be at (x, y) with $y > 0$.

Case (i): $|AC| = |AB|$. In this case, C is on a circle with center A and radius 1. We have that $x^2 + y^2 = 1$ and

$$2 - 2x = x^2 + y^2 - 2x + 1 = (x-1)^2 + y^2 = \frac{1}{k^2}.$$

Since the area is maximized when $AC \perp AB$, we must have $x = 0$ and $k = 1/\sqrt{2}$.

Case (ii): $|BC| = |AB|$. In this case, C is on a circle with center B and radius 1. We have that $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = k^2$, so that $2x = k^2$. In this case, the area is maximized when $BC \perp AB$, so that $x = 1$ and $k = \sqrt{2}$. □

Comment. In this problem, we are supposed to perform the optimization for each fixed value of k, rather than, as the solution indicates, looking at possible cases of isosceles triangles and finding a maximizing value of k.

When $k \neq 1$, the locus of C is a circle of Apollonius whose center is on the line AB and which lies to one side of the perpendicular bisector of AB. With the coordinates of A, B, C respectively $(0,0)$, $(1,0)$, (x,y), the locus of C is given by the equation

$$x^2 + y^2 = k^2[(x-1)^2 + y^2]$$

or

$$(k^2-1)x^2 - 2k^2x + (k^2-1)y^2 + k^2 = 0.$$

For example, when $k = \sqrt{2}$, the locus is the circle with equation $(x-2)^2 + y^2 = 2$. This has center $(2,0)$, radius $\sqrt{2}$, and $x-$intercepts $(2-\sqrt{2},0)$ and $(2+\sqrt{2},0)$. The area of triangle ABC is maximized when C is at $(2, \pm\sqrt{2})$; in this case, the triangle of maximum area is not isosceles.

However, there is a possibility of an isosceles maximizing triangle when the circle of radius 1 and center A or B intersects the circle of Apollonius at the endpoint of its vertical diameter. This happens when $k = \sqrt{3}$, when the equation of the locus of C is $(x-\frac{3}{2})^2 + y^2 = \frac{3}{4}$ and the area of triangle ABC is maximized when C is at $(\frac{3}{2}, \pm\frac{\sqrt{3}}{2})$ and $AB = BC$. It also happens when $k = 1/\sqrt{3}$, when the locus of C has equation $(x+\frac{1}{2})^2 + y^2 = \frac{3}{4}$ and the maximizing position of C is at $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$.

3.6 A triangle condition

Problem. Let s, r, R, respectively, be the semiperimeter, inradius and circumradius of a triangle ABC with sides a, b, c. Determine a necessary and sufficient condition on s, r, R that the sides a, b, c be in arithmetic progression.

Solution (by a student). We have that $4Rrs = abc$ and $2s = a+b+c$ for any triangle. The condition that the sides are in arithmetic progression

is that $a + c = 2b$. Thus, we arrive at the conditions that $a + c = (4s)/3$ and $ac = 6Rr$, so that a and c are solutions of the quadratic equation

$$x^2 - \frac{4}{3}sx + 6Rr = 0.$$

This is solvable if and only if its discriminant is positive, *i.e.*, $2s^2 - 27Rr \geq 0$. This is the desired condition. □

Comments. The condition actually holds for any triangle, as can be seen from an application of the arithmetic-geometric means inequality $a + b + c \geq 3\sqrt[3]{abc}$. In fact, if we pose the analogous problem for the sides to be in geometric progression, we have that $b = \sqrt[3]{4Rrs}$ and a and c are the solutions to the quadratic equation

$$x^2 - (2s - \sqrt[3]{4Rrs})x + \sqrt[3]{(4Rrs)^2} = 0.$$

Applying the discriminant condition for solvability yields that

$$\begin{aligned} 0 &\leq (2s - \sqrt[3]{4Rrs})^2 - (2\sqrt[3]{4Rrs})^2 \\ &= (2s + \sqrt[3]{4Rrs})(2s - 3\sqrt[3]{4Rrs}). \end{aligned}$$

Since the first factor of the right side is always positive, we arrive at exactly the same condition as for the arithmetic progression case.

So what is wrong? One way to get into the situation is to look at an example. Since R, r and s are easy to determine for right triangles, consider the triangle for which $(a, b, c) = (5, 12, 13)$. Then $(R, r, s) = (13/2, 2, 15)$ and the condition $2s^2 \geq 27Rr$ is clearly satisfied. We are led to the quadratic equation $x^2 - 20x + 78 = 0$. The two solutions of this equation are $10 - \sqrt{22}$ and $10 + \sqrt{22}$. If we take these as our values of a and c and use either $b = (4Rrs)/(ac)$ or $b = 2s - (a + c)$, we find that $b = 10$.

However, when $(a, b, c) = (10 - \sqrt{22}, 10, 10 + \sqrt{22})$, the corresponding values of the parameters are given by $(R, r, s) = (13, 1, 15)$, these yielding the same quadratic as before.

For any triangle (a, b, c), we can form the quadratic equation $x^2 - \frac{4}{3}sx + 6Rr = 0$. However, a and c are roots of this equation if and only if $ac = 6Rr$ and $a + c = \frac{4}{3}s$, and these conditions require $3b = 2s$. When this fails, the quadratic has two other roots a' and c' for which $a' + c' = \frac{4}{3}s$ and $a'c' = 6Rr$. If we take $b' = 2s - (a' + c') = \frac{2}{3}s$, then $4Rrs = \frac{2}{3}(a'c')(\frac{3}{2}b') = a'b'c'$. However, in this case, while s is the semiperimeter of triangle (a', b', c') and Rr is the product of the circumradius and the

inradius, the actual circumradius and inradius differ from R and r, respectively.

This problem caught the interest of Ken McCaffrey. The condition given in the "solution" applied to any triangle, and he was concerned that the examples started with a triangle whose sides were not in arithmetic (or geometric) progression and ended up by producing a triangle with sides in progression which led to the same quadratic equation.

He took an approach that avoided the quadratic equation, whose use, in his opinion, violated the nature of the question posed. He starts with the *product* $6Rr$ and picks values of a and c for which $6Rr = ac$. There are infinitely many possibilities; while the values of the circumradius and inradius may differ from the given R and r, the product remains the same. For example, take the product 78; if $(a, c) = (6, 13)$, then (for sides in arithmetic progression), $b = 9.5$ and $s = 14.25$, and for $(a, c) = (10 - \sqrt{22}, 10 + \sqrt{22})$, then $b = 10$ and $s = 15$. However, not every choice of (a, c) will lead to a triangle, such as $(a, c) = (3, 26)$. To ensure that we have a triangle, assuming $a < c$ and $2b = a + c$, we require that $c < 3a$. With $p = 6Rr = ac$, this leads to $\sqrt{p/3} < a < p/\sqrt{p/3}$.

He concludes, "in summary, the conditions turn out to be as follows:

- for a given product p, select a and c such that $ac = p$ and $\sqrt{p/3} < a < p/\sqrt{p/3}$;
- $b = (a + c)/2$;
- $s = (a + b + c)/2$;
- $r = (bc \sin A)(a + b + c)$ or $r = (bc \sin A)/(2s)$;
- $R = ac/(6r)$."

For the record, here is a sketch of a correct solution to the problem. First establish, for any triangle, that $a+b+c = 2s, ab+bc+ca = s^2+4Rr+r^2$ and $abc = 4Rrs$, where a, b, c are the side lengths, R is the circumradius, r is the inradius and s is the semi-perimeter. Then a, b, c are in arithmetic progression if and only if

$$\begin{aligned} 0 &= (b + c - 2a)(c + a - 2b)(a + b - 2c) \\ &= (2s - 3a)(2s - 3b)(2s - 3c) = 2s(s^2 - 18Rr + 9r^2), \end{aligned}$$

or more simply $s^2 = 9r(2R-r)$. Also, a, b, c are in geometric progression if and only if

$$\begin{aligned} 0 &= (a^2 - bc)(b^2 - ac)(c^2 - ab) \\ &= 32Rrs^4 - (s^2 + 4Rr + r^2)^3. \end{aligned}$$

My opinion was that the question of the existence of the triangle does not arise, since the problem presumes that a triangle is given with sides (a, b, c) from which R, r, s can be calculated; it then seeks conditions on these calculated values that in fact the original triangle has sides in arithmetic or geometric progression. The quadratic equation was discussed in the faulty solution, and so should figure in the analysis of the solution. I sent McCaffrey the solution in the last paragraph. This drew the response, "The existence question in my mind is that just because you have a triangle (a, b, c) and of course (R, r, s) can be calculated, it does not necessarily follow that using the information it can be presumed that an arithmetic or geometric sided triangle exists. Furthermore, since the original sides were not in [arithmetic] or [geometric] progression, and R, r, s will eventually be calculated to be something other than the original, of just what purpose [were] the original triangle's dimensions [given]? As it turns out, my solution demonstrates that for any given (R, r), many (a, b, c) can be determined. So, in fact an infinite number of such triangles exist."

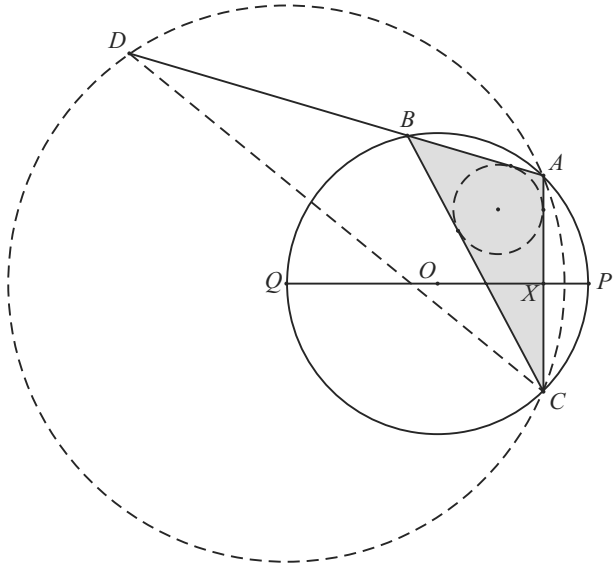
McCaffrey poses an interesting problem. Suppose, for a given triangle, the circumcircle and incircle are drawn. Now erase the triangle. What restrictions are there on the placement of the smaller circle inside the larger one? Clearly, it cannot just be any circle inside the larger one. If the two circles are concentric, must the triangle be equilateral?

Paul Yiu observes that, in fact, Euler has given us a condition that these circles must satisfy. If d is the distance between the centers of the circumcircle and incircle of a triangle, then $d^2 = R(R - 2r)$. A proof of this can be found on page 29 of the book *Geometry Revisited* by H.S.M. Coxeter and S. Greitzer (MAA, 1967). The proof also shows that, given circles of radii R and r with centers d apart and an arbitrary chord of the outer circle tangent to the inner one, the tangents from the endpoints of the chord to the inner circle intersect at a point on the outer circle. It is known that a triangle is equilateral if and only if its circumcenter and incenter coincide.

For any triangle, Euler's equation affirms that $R \geq 2r$. Yiu provides a straightedge and compasses construction that, given any pair R and r of radii, with $R \geq 2r$, and a circle of center O and radius R, we can find within it a circle of radius r and a triangle with sides in arithmetic progression for which these are the circumcircle and incircle.

Let PQ be any diameter of the circle with center O and radius R; mark a point on it for which $PX = r$. Construct the chord AC perpendicular to PQ at X. The length of AC is equal to $2\sqrt{r(2R - r)}$. Construct the circle with center Q that passes through A, and let AD be a chord of the latter

circle whose length is $2 \cdot AC$; suppose that it intersects the given circle at B. Since $\angle ABC = \angle AQC = 2\angle ADC$, $\angle BDC = \angle ADC = \angle BCD$ and $BC = BD$. Triangle ABC has circumradius R, semiperimeter s and sidelengths in arithmetic progression. Its inradius must be r.



3.7 There are no isosceles triangles

Proposition. Let A, B, C be three noncollinear points of the plane for which $AB = AC$. Then the arc BC of the circumcircle of triangle ABC that does not contain A is contained in the right bisector of the side BC.

Proof. Let P be any point on the arc BC opposite A. Since AB and AC are equal chords of the circumcircle of ABC, they subtend equal angles at P, so that $\angle APB = \angle APC$.

A rotation about the center of the circle that carries A to B must carry C to A and P to some point T on the circle. Thus, triangle APC gets carried to triangle BTA.

Since $AP = BT$, these chords must subtend equal angles at the circumference of the circle, so that $\angle PBA = \angle BAT = \angle ACP$. Consider triangles ABP and ACP. Since $\angle APB = \angle APC$ and $\angle PBA = \angle PCA$, it follows that $\angle BAP = \angle PAC$, so that P must lie on the right bisector of BC. □

Corollary. *There are no nondegenerate isosceles triangles.*

Proof. From the above proposition, we can deduce that B and C on the arc of the circumcircle must be on the right bisector of BC. But this means that they must coincide. □

Comment. AP is indeed equal to BT, but equal chords can also subtend angles that are supplementary. If we look carefully at the rotation, we see that this is the case here. The image of B under the rotation winds up on the opposite side of BT than A, so that $\angle PBA + \angle BAT = 180°$ (consistent with $\angle PBA + \angle PCA = 180°$).

3.8 Maximizing an area

Problem. The area of a triangle ABC is 1. Trapezoid $DEFG$ is constructed so that G and F lie on the base BC in this order, D lies on AB, E lies on AC, $DE \| BC$, $EF \| AB$ and $DG \| AC$. Determine the maximum possible area of trapezoid $DEFG$.

Solution. Let s and t be positive reals for which $s + t = 1$ and $AD : DB = s : t$. Then the area of triangle ADE is s^2 and the common area of the congruent triangles DBG and EFC is t^2. The area of the trapezoid is $1 - s^2 - 2t^2$.

We have that

$$\begin{aligned} 1 - s^2 - 2t^2 &= (s + t)^2 - s^2 - 2t^2 = 2st - t^2 \\ &= s^2 - (s - t)^2 \leq s^2. \end{aligned}$$

Equality occurs if and only if $s = t = \frac{1}{2}$. Thus, the maximum area is $\frac{1}{4}$. □

However, when $s = \frac{2}{3}$, we find that the area of the trapezoid is $\frac{1}{3}$. The solution was supplied by a competitor to a problem on the 2007 *Euclid Contest* operated by the Canadian Mathematics Competition.

Comment. In order that the segments DG and EF do not cross, we require that $s \geq \frac{1}{2}$. The area of the trapezoid is

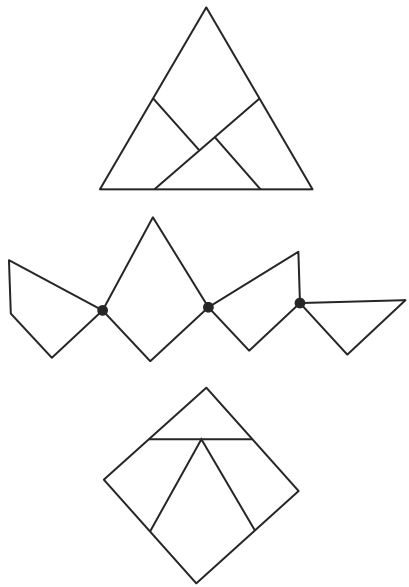
$$1 - s^2 - 2t^2 = \frac{1}{3} - 3\left(s - \frac{2}{3}\right)^2$$

and this is maximum when $s = 2/3$. In the purported solution, s and t are not independent. Altering either term of $s^2 - (s - t)^2$ changes both, so that reducing the size of $(s - t)^2$ may be at the expense of increasing s^2 and so the whole difference.

3.9 Making a square out of a triangle

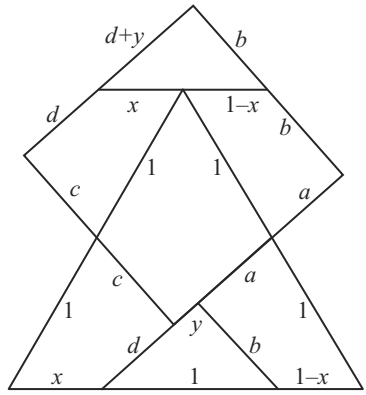
Cambridge University Press has recently reprinted in softcover the 1997 book, *Dissections: Plane and Fancy*, by Greg N. Frederickson (ISBN 0-521-52582-9). It provides an excellent comprehensive survey of the art and science of dissecting polygons and rearranging the pieces to form other polygons or sets of polygons.

Chapter 12, provocatively entitled, *Strip Teased*, opens with a famous solution to a puzzle posed in 1902 by Henry Dudeney, namely to dissect an equilateral triangle into pieces that can be rearranged into a square (pages 136–137). Here is the illustration of the solution. Although the details are not specified, one might suppose from the diagram that the base of the triangle is divided so that the end segments are equal to one quarter of the base. But, if this is so, do the pieces really constitute a square? □



If we assign to the side of the triangle the length 2, then we can mark in various lengths. I have superimposed the triangle and square in the diagram below.

It is clear that the pieces rearrange to form a rectangle, so that $b = c$ and $a = d$. The rectangle will be a square if and only if $2b = 2c = 2d + y = 2a + y$, so that $b + c = a + d + y$. What implication does this have for x? Certainly x cannot be equal to $1/2$. Otherwise, the midline of the triangle parallel to the base and the segment of length 1 of the base would



be opposite sides of a rectangle; $b + c$ could not be equal to the diagonal $a + d + y$ of the rectangle.

So the midline and segment are the opposite edges of a parallelogram of area $\sqrt{3}/2 = b(2a + y) = 2b^2$, whence $b = 3^{1/4}/2 \sim 0.658037$ and $2a + y = 3^{1/4} \sim 1.316074$. It is a straightforward exercise to verify that $2a + y$ is the length of the shorter diagonal and that x is approximately 0.509015.

Greg Frederickson, author of *Dissections: Plane & Fancy*, notes that in his second book, *Hinged dissections: swinging & twisting* (Cambridge University Press, 2002), he did mention on page 269 that the base of the triangle was not split in the ratio 1:2:1, "Actually, in 1984, Don Crowe and Isaac Schoenberg pointed out that such a claim (made in the 1969 edition of Steinhaus's book) was wrong." He adds that,

> Three years ago, I posted angles and lengths for the pieces on my webpage: `www.cs.purdue.edu/homes/gnf/book2/trisqu.html` In contrast to your Figure 2, I took the side of the square to be 1.
>
> When I taught ninth grade mathematics in Baltimore many years ago, the only halfway-decent bulletin board that I ever made showed the dissection of two pentagrams to a 10-pointed star that was in Harry Lindgren's book.

3.10 Ptolemy's inequality in space

Omicron: You seem to be very busy.

Upsilon: I am trying to solve the following problem:

Let $ABCD$ be a tetrahedron. Prove that

$$AB \cdot CD + AC \cdot BD \geq AD \cdot BC \ .$$

Omicron: Oh, the solution is quite straightforward. Let me call the three products in the equation c, b, a respectively, for brevity. So the required inequality is $c+b \geq a$. By symmetry, it is equivalent to prove that $b+a \geq c$ or that $a+c \geq b$. We solve the problem by a contradiction argument. If the result is false, then we must have $c+b < a$, $b+a < c$ and $a+c < b$. Adding these gives $2(b+c+a) < (b+c+a)$, which is false. Therefore the result follows

Upsilon: Something seems fishy.

Omicron: What is the difficulty? If we relabel the tetrahedron, interchanging A and C, and follow the argument we used to get $c+b \geq a$, then we get $a+b \geq c$, and if we prove the latter, then we get the former. So the two inequalities stand or fall together. Likewise for the other one as well. So the only way for one of them to be false is for the three of them to be false.

Upsilon: But you are presuming a proof for $c+b \geq a$, and I will agree that if such a proof is adduced, you can go through it changing the notation to get one of the other two inequalities. However, your process is introducing a kind of circularity into the situation. For what you are offering is a proof, and you are not entitled to depend on the existence of a proof to make your proof work. You could apply the same argument to the sides of a triangle to get the triangle inequalities.

Omicron: So what?

Upsilon: Let us look at what the result is really saying. If you form the three products of opposite edges of a tetrahedron, then any one of them does not exceed the sum of the other two. If you want to argue by contradiction, then you need only assume that one of them fails. Likewise for the triangle inequalities.

Omicron: But it is not possible for one inequality to fail without all of them failing, as I indicated before.

Upsilon: Look at the three numbers, a, b, c. One of these numbers is maximum, say a. So regardless of what the numbers might be, you are always going to get two of the inequalities $b+a > c$ and $a+c > b$ for nothing. The only one you have to work for (and the only one that might fail) is $b+c > a$. Since in the original tetrahedron, you are not given which product is largest, you need a proof strong enough to cover this case.

Omicron: You are just envious because I have a neat argument that does not depend on knowing anything about tetrahedra.

3.11 The area of a cross-section

Problem. The base of a pyramid $ABCDV$ is a rectangle $ABCD$ with $|AB| = a$, $|BC| = b$ and $|VA| = |VB| = |VC| = |VD| = c$. Determine the area of the intersection of the pyramid and the plane parallel to the edge VA that contains the diagonal BD.

Solution. Let O be the center of the rectangle and M the midpoint of VC; VO is perpendicular to the rectangle $ABCD$. The dilation with center C and factor $1/2$ takes the segment VA to the segment MO, so that $MO \| VA$. Hence the plane in question contains M, B and D and the intersection is a triangle MBD with base BD and altitude MO. Therefore the desired area is

$$\frac{1}{2}(\frac{1}{2}c)\sqrt{a^2+b^2} = \frac{c}{4}\sqrt{a^2+b^2}. \quad \square$$

Comment. When $a \neq b$, MO is not the altitude of triangle BMD. Since BM and DM are the respective medians of triangle BVC with sides b, c, c and DVC with sides a, c, c, we find that $|BM| = \frac{1}{2}\sqrt{2b^2+c^2}$, $|DM| = \frac{1}{2}\sqrt{2a^2+c^2}$. Thus triangle MBD is not generally isosceles when $a \neq b$ and its altitudes are not medians.

Another way to see that V and M are not on the same plane orthogonal to BD is to note that V is on the plane through O orthogonal to BD while C (and hence M) is not.

To obtain the correct answer, let $\theta = \angle BMD$. Then, by the law of Cosines,

$$\cos\theta = \frac{c^2 - a^2 - b^2}{\sqrt{2b^2+c^2}\sqrt{2a^2+c^2}},$$

whence

$$\sin\theta = \frac{\sqrt{4c^2(a^2+b^2) - (a^2-b^2)^2}}{\sqrt{2b^2+c^2}\sqrt{2a^2+c^2}}.$$

The required area is

$$\frac{1}{2}|BM||DM|\sin\theta = \frac{1}{8}\sqrt{4c^2(a^2+b^2) - (a^2-b^2)^2}.$$

3.12 Clipping the corners off

Problem. Let **C** be a cube with edges of length 2. Construct a solid figure with fourteen faces by cutting off with planes perpendicular to the body diagonals all eight corners of **C**, keeping the newly formed faces identical. If the fourteen faces formed all have the same area, determine the common area of the faces.

Solution (by a student). Let x be the distance from a vertex of the cube to the nearest point on an adjacent edge through which the cut is made. The base of the corner removed is an equilateral triangle of side $x\sqrt{2}$ and area $x^2\sqrt{3}/2$. Each face of the cube has had removed from it four isosceles right triangles with hypotenuse $x\sqrt{2}$ to create an octagonal face of the solid figure; this face has area $4-2x^2$. Thus, we require that $x^2\sqrt{3}/2 = 4-2x^2$, or $x^2 = 8(4-\sqrt{3})/13$. The common area of the faces of the solid figure is $(16\sqrt{3}-12)/13$. □

However, is this value of x no greater than 1?

3.13 Slippery Centroids

The following essay was submitted by John M. Alongi and Steve Kennedy.

> The centroid of a triangular lamina with uniform density lies at the intersection of the medians of its boundary triangle (see [2]). Let A, B and C be vertices of a triangular lamina ABC with centroid D. Choose points A', B' and C' on the medians of ΔABC so that $\Delta A'B'C'$ is similar to ΔABC. Then the medians of ΔABC contain the medians of $\Delta A'B'C'$ and, therefore, contain the centroid at D. Thus the triangular annular lamina obtained by excising the lamina $A'B'C'$ from the lamina ABC has centroid at D. This is true for any A', B' and C' chosen as above. Let A', B', C' converge to A, B, C, respectively, along the medians of ΔABC while maintaining the similarity of ΔABC to $\Delta A'B'C'$. The centroids of the corresponding annular laminae always lie at D. Consequently, the limit of these annular laminae — a one-dimensional triangular wire with uniform density — also has its centroid at the intersection of the medians of ΔABC.

Resolution. The centroid of a one-dimensional triangle with uniform density lies at the intersection of the angle bisectors of the medial trian-

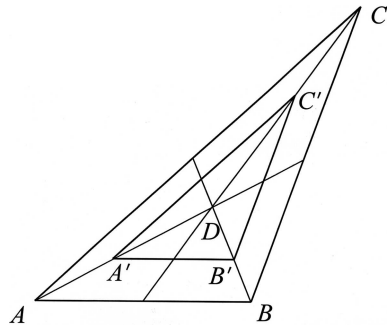


Figure 1.

gle (see [1]). This is not, in general, the intersection of the medians. Alternatively, the centroid of a triangle is a weighted average of the centroids of its edges. If these edges are uniformly dense, then these weights are proportional to the sidelengths of the triangle.The centroid of the triangular annular lamina constructed above with the boundary triangles ΔABC and $\Delta A'B'C'$ is the weighted average of the centroids of the trapezoids $AA'B'B$, $BB'C'C$ and $CC'A'A$. As a consequence of uniform density, these weights are proportional to the areas of the trapezoids. Thus, the above argument is correct when these trapezoids all have the same height, since the height factor occurs in the area of each trapezoid *and* in the total area of the triangular annular lamina. That is, after cancelling this common factor, the weights are proportional to the average widths of the trapezoids. If the trapezoids have equal heights, then it can be shown that the centroids of the annular laminae converge to the intersection of the angle bisectors of the boundary's medial triangle.

As constructed, however, the trapezoids do not necessarily have equal heights. The construction yields trapezoids with heights h, ah/b, and ah/c, on sides of length a, b and c, respectively (Figure 2). In the limit, the choice of trapezoids induces a mass distribution on the triangular wire so that the ratio of the masses of any two edges is equal to the limiting ratio of the areas of the corresponding trapezoids. In the above construction, the particular choice of trapezoid yields, in the limit, a triangle with edges of identical mass. Equivalently, the centroid of a triangular wire with edges of identical mass and with the centroid of each edge located at the edge's midpoint lies at the intersection of the medians. In fact, by appropriately choosing the heights of the trapezoids, it is possible, in the limit, to impose a mass distribution on the edges of the limit triangle yielding a centroid at any desired location in or on the medial triangle. More generally, by considering

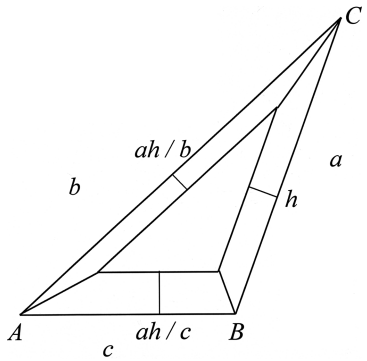


Figure 2.

regions other than trapezoids, it is possible to achieve any desired mass distribution on the edges, and thus realize any center of mass in or on the triangle.

In short, the flaw in the original argument lies in its assertion that the limit of the uniformly dense triangular annular laminae is a uniformly dense one-dimensional triangular wire. One can make sense of passing to the limit, but the limit does not preserve uniform density. Quite the opposite — passage to the limit can impose any desired density on the edges of the triangle.

Editor's comment. Readers might wish to compare this note with the item on the lopsided uniform rod [3]. The authors note that their resolution can be readily modified to this problem as well.

References

[1] Tom Apostol & Mamikon Mnatsakanian, Finding Centroids the Easy Way, *Math Horizons*, September, 2000.

[2] David Halliday & Robert Resnick, *Physics*. New York: John Wiley & Sons, 1978.

[3] The Lopsided Uniform Rod, *College Mathematics Journal* 23 (1992) 36–37; 24 (1993) 76.

Note

In Chapter 8, item 5, there is a discussion of the probability that an arbitrary triangle is obtuse.

CHAPTER 4

Limits, Sequences and Series

4.1 Not many real sets

Theorem. *The cardinality of the class of subsets of the closed unit interval does not exceed the cardinality of the closed unit interval itself.*

Proof. Let S be a subset of $[0, 1]$. We assign to S an element of $[0, 1]$, written in binary form as follows. If S is void, assign to S the number $0.00000\ldots$. Otherwise, the first binary digit after the decimal point of the corresponding number is 1. If S has nonvoid intersection with the interval $[0, 1/2]$, let the second binary digit of the corresponding number be 1; otherwise, assign 0. If S has nonvoid intersection with the interval $[1/2, 1]$, let the third binary digit be 1; otherwise, assign 0.

In a similar way we assign the next four digits of the number corresponding to S, according as to whether S intersects the four equal intervals into which $[0, 1]$ is partitioned, and then the next eight digits, and so on.

Each point in S is uniquely determined as the intersection of a nested sequence of intervals of the form $[(i-1)2^{-j}, i2^{-j}]$, and so is tracked by a 1 in the appropriate places of the binary expansion of the number corresponding to S. Thus, we obtain a correspondence between the class of sets S and some subset of $[0, 1]$. This yields the desired result. □

Comment. Compare the number that corresponds to the set of rationals to the one corresponding to the set of nonrationals.

4.2 A wrong version of Stirling's formula

Keith Brandt invites students and teachers to look more closely at the limits underlying statements of the form $f(n) \approx g(n)$. He uses Stirling's formula

as a case in point. This states that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\cdot n^n \cdot e^{-n}} = 1,$$

so that, for large n, we may assume $n! \approx \sqrt{2\pi n}\cdot n^n \cdot e^{-n}$. A colleague of his, a chemist, once showed him how to use Stirling's formula and logarithms to derive a weaker result known as Stirling's approximation (often used in statistical mechanics). This is derived as follows.

Apply the logarithm to both sides of Stirling's formula to obtain

$$\ln(n!) \approx \frac{1}{2}\ln(2\pi n) + n\ln n - n.$$

The term $\frac{1}{2}\ln(2\pi n)$ is then declared "insignificant" and neglected to yield Stirling's approximation

$$\ln n! \approx n\ln n - n. \tag{1}$$

While this argument may be intuitive, it is not very satisfying from a mathematician's point of view. First, Stirling's approximation can be derived from scratch using only elementary calculus. Use Riemann sums to show that

$$\int_1^n \ln x\,dx \le \ln n! \le \int_2^{n+1} \ln x\,dx.$$

Evaluate the integrals, divide by $n\ln n - n$ and take limits. Second, the connection between Stirling's formula and Stirling's approximation is a bit subtle, and deserves a closer look.

The chemist's approach relies on the following two results, the first of which justifies applying the logarithm to both sides of Stirling's formula.

Lemma 1. *Suppose* $\lim_{n\to\infty} f(n) = +\infty$ *and* $\lim_{n\to\infty} g(n) = \infty$. *If* $\lim_{n\to\infty} f(n)/g(n) = K > 0$, *then* $\lim_{n\to\infty}(\ln f(n))/(\ln g(n)) = 1$. *In particular, if* $f(n) \approx g(n)$, *then* $\ln f(n) \approx \ln g(n)$.

Lemma 2. $\lim_{n\to\infty}(\ln n)/(\ln n!) = 0$.

Lemma 1 is a good exercise for students in advanced calculus and Lemma 2 for students in elementary calculus. Lemma 1 suggests a second question: can we exponentiate both sides of $f(n) \approx g(n)$? Doing so to Stirling's approximation (1) yields

$$n! \approx \frac{n^n}{e^n}.$$

This elegant looking formula is not valid, since $\lim_{n\to\infty} n!/(n^n e^{-n}) = +\infty$. It is, however, mentioned in the literature (see [1, p. 24] or [2, p. 263]). Of course, ter Haar immediately applies the wrong formula to an expression involving the logarithm of $n!$. Franklin goes on to sharpen his "rough estimate" until he reaches Stirling's formula. How many students are aware of Lemma 1? How many are aware that there is no similar result for exponentiation?

References

[1] ter Haar, D., *Elements of statistical mechanics*, Rinehart, 1954.

[2] Frankin, P. *Methods of advanced calculus*, McGraw-Hill, 1944.

4.3 Hospitalization I

Michael Caulfield asked his students for the interval of convergence for the (unknown) power series expansion of the rational function

$$f(x) = \frac{7x-1}{3x^2+2x-1}.$$

One of his students apparently thought, "why use partial fractions when you can simply use l'Hôpital's Rule?" He used l'Hôpital's Rule to convert $f(x)$ to

$$\frac{7}{6x+2} = \left(\frac{7}{2}\right)\frac{1}{3x+1} = \left(\frac{7}{2}\right)\frac{1}{1-(-3x)}.$$

He moved on to the geometric series representation $\frac{7}{2}\sum_{n=0}^{\infty}(-3x)^n$, which he identified as converging if $|-3x| < 1 \Longrightarrow |x| < \frac{1}{3}$. This, of course, is the correct answer.

4.4 Hospitalization II

Bill Sands received the following solution from a first year calculus student.

Problem. Find

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2}.$$

Solution.

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \lim_{x\to 0}\frac{\sin 2x}{x^2} = \lim_{x\to 0}\frac{\sin x}{x} + \lim_{x\to 0}\frac{\sin x}{x} = 1+1 = 2.$$

4.5 L'Hôpital's Circle

Gary Kennedy submitted this attempt by a student to evaluate

$$\lim_{x\to\infty}\left(\frac{x}{x^2+1}+\arctan x\right).$$

Putting the expression over a common denominator and applying L'Hôpital"s Rule twice gives

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x}{x^2+1}+\arctan x\right) &= \lim_{x\to\infty}\frac{x+(x^2+1)\arctan x}{x^2+1}\\ &= \lim_{x\to\infty}\frac{1+1+2x\arctan x}{2x}\\ &= \lim_{x\to\infty}\frac{1+x\arctan x}{x}\\ &= \lim_{x\to\infty}\frac{\frac{x}{x^2+1}+\arctan x}{1}\\ &= \lim_{x\to\infty}\left(\frac{x}{x^2+1}+\arctan x\right).\end{aligned}$$

This seems to be a clear case of a student so wrapped up in the application of a technique that a straightforward solution is completely overlooked.

4.6 A proof that $-1 = 1$

Sung Soo Kim of Hanyang University in Ansan, Kyunggi, Korea, submitted the following application of l'Hôpital's Rule:

$$\begin{aligned}1 &= \lim_{x\to\infty}\frac{x-\sin x^2}{x+\sin x^2} = \lim_{x\to\infty}\frac{1-2x\cos x^2}{1+2x\cos x^2}\\ &= \lim_{x\to\infty}\frac{1/x-2\cos x^2}{1/x+2\cos x^2} = \lim_{x\to\infty}\frac{-2\cos x^2}{2\cos x^2} = -1.\end{aligned}$$

4.7 A limit at $-\infty$

Danrun Huang reports that most students of his calculus class seemed to be satisfied with the following solution to the problem of determining $\lim_{x\to-\infty}(x+\sqrt{x^2+2x})$. The problem (not the solution) came from the text *Single variable calculus: early transcendentals* (4th edition), by James Stewart (Brooks/Cole, 1999; Section 2.6, #22).

After rationalizing the numerator, we find that the limit is equal to

$$\begin{aligned}\lim_{x\to-\infty}\frac{-2x}{x-\sqrt{x^2+2x}} &= \lim_{x\to-\infty}\frac{-2}{1-\frac{\sqrt{x^2+2x}}{x}}\\ &= \lim_{x\to-\infty}\frac{-2}{1-\sqrt{\frac{x^2+2x}{x^2}}}\\ &= \lim_{x\to-\infty}\frac{-2}{1-\sqrt{1+\frac{2}{x}}} = -\infty.\end{aligned}$$

Then one student indicated that his graphing calculator showed that the limit looked like -1. So Huang asked every student in the class to check and judge which answer was right, reminding them that sometimes a calculator could lie. With his help, the student eventually found the hidden "bug" in the argument and learned an important lesson on how to put a negative variable inside a square root. The calculator was honest this time!

4.8 Finding an asymptote

Carl Libis sends in a student solution to the exercise of finding all horizontal asymptotes for the graph of

$$f(x) = \frac{2}{x-3} - \frac{x}{x+2}.$$

$$\begin{aligned}f(x) = \frac{2}{x-3} - \frac{x}{x+2} &= \frac{2(x+2)}{(x-3)(x+2)} - \frac{x}{x+2}\\ &= \frac{2x+4}{x^2-x-6} + \frac{-x}{x+2} = \frac{x+4}{x^2-4}\end{aligned}$$

(presumably by adding the numerators and the denominators). Thus

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{x+4}{x^2-4} = \frac{0+4}{0-4} = -1,$$

so the horizontal asymptote is $y = -1$ (which is correct).

4.9 Series that converges and diverges

Consider the infinite series

$$\sum_{n=2}^{\infty}(-1)^n \ln\left(1-\frac{1}{n}\right).$$

This series can be shown to converge by the alternating series test. But we also have

$$\begin{aligned}
&\sum_{n=2}^{\infty}(-1)^n \ln\left(1-\frac{1}{n}\right) \\
&\qquad = \sum_{n=2}^{\infty}(-1)^n \ln\left(\frac{n-1}{n}\right) = \sum_{n=2}^{\infty}(-1)^n[\ln(n-1)-\ln n] \\
&\qquad = [\ln 1 - \ln 2] - [\ln 2 - \ln 3] + [\ln 3 - \ln 4] - [\ln 4 - \ln 5] + \cdots \\
&\qquad = -2\ln 2 + 2\ln 3 - 2\ln 4 + 2\ln 5 - 2\ln 6 + \cdots \\
&\qquad = \sum_{n=2}^{\infty}(-1)^{n+1}(2\ln n).
\end{aligned}$$

This last series diverges since the nth term does not tend to zero. $\diamond$

Consider also these treatments of the alternating harmonic series. First, we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} \\
&\qquad = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots \\
&\qquad = 1 + \left(\frac{1}{2} - 1\right) + \left(1 - \frac{2}{3}\right) + \left(\frac{3}{4} - 1\right) + \cdots \\
&\qquad\quad + \left(\frac{2n-1}{2n} - 1\right) + \left(1 - \frac{2n}{2n+1}\right) + \cdots \\
&\qquad = 1 + \frac{1}{2} + (-1+1) - \frac{2}{3} + \frac{3}{4} + (-1+1) - \cdots \\
&\qquad\quad + \frac{2n-1}{2n} + (-1+1) - \frac{2n}{2n+1} - \cdots \\
&\qquad = 1 + \left(\frac{1}{2} - \frac{2}{3}\right) + \left(\frac{3}{4} - \frac{4}{5}\right) + \cdots + \left(\frac{2n-1}{2n} - \frac{2n}{2n+1}\right) + \cdots \\
&\qquad = 1 + \left(-\frac{1}{6}\right) + \left(-\frac{1}{20}\right) + \cdots + \left(\frac{4n^2 - 1 - 4n^2}{4n^2 + 2n}\right) + \cdots \\
&\qquad = 1 - \sum_{n=1}^{\infty}\frac{1}{2n(2n+1)}
\end{aligned}$$

which is absolutely convergent.

On the other hand,

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots\\ &= 1+\left(\frac{1}{2}-1\right)+\frac{1}{3}+\left(\frac{3}{4}-1\right)+\frac{1}{5}+\cdots\\ &\quad+\left(\frac{2n-1}{2n}-1\right)+\frac{1}{2n+1}+\cdots\\ &= 1+\frac{1}{2}+\left(-1+\frac{1}{3}\right)+\frac{3}{4}+\left(-1+\frac{1}{5}\right)+\frac{5}{6}+\cdots\\ &\quad+\frac{2n-1}{2n}+\left(-1+\frac{1}{2n+1}\right)+\cdots\\ &= 1+\frac{1}{2}-\frac{2}{3}+\frac{3}{4}-\frac{4}{5}+\cdots+\frac{2n-1}{2n}-\frac{2n}{2n+1}+\\ &= 1+\sum_{n=2}^{\infty}(-1)^n\frac{n-1}{n}\end{aligned}$$

which is divergent.

More generally, any alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$, with $a_n > 0$, can be manipulated to a divergent series $\sum_{n=1}^{\infty}(-1)^{n+1}(1-a_n)$ by replacing each oddly indexed term $-a_{2k+1}$ by $(1-a_{2k+1})-1$ and rebracketing.

Contributed by Doug Kuhlman and Ollie Nanyes.

4.10 When the limit comparison test fails

Franciszek Prus-Wiśniowski invites us to consider the following result:

$$\sum_{n=1}^{\infty}\frac{a_n}{n}\text{ converges }\iff\sum_{n=1}^{\infty}\frac{a_n}{n+1}\text{ converges.}$$

He continues: Is it false or true? Or does it depend on the sequence $\{a_n\}$? This equivalence was used in the course of the proof of Theorem 3 in [2]. It is tempting to prove it using the Limit Comparison Test, a form of which is the following [1, Thm. 11.6.3]:

Limit Comparison Test. *If $\sum b_n$ and $\sum c_n$ are series with positive terms and if the sequence of quotients $\{b_n/c_n\}$ has a positive limit, then either both series converge or both diverge.*

Taking $b_n = a_n/n$ and $c_n = a_n/(n+1)$, we get that

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1,$$

so the equivalence of the convergence of the two sequences is true by the Limit Comparison Test. But, what about the assumption that b_n and c_n should be positive. It seems, after some thought, that the test might work even for series whose terms are not necessarily all positive, since if b_n/c_n tends to a positive number, then b_n and c_n should eventually have the same sign, and we can come to the desired conclusion after disregarding an initial number of terms.

However, an example shows that this reasoning is false.

Define

$$b_n = (-1)^n \left(\frac{1}{\sqrt{n}} + \frac{(-1)^n}{n} \right)$$

and

$$c_n = \frac{(-1)^n}{\sqrt{n}}$$

for all positive integers n. We have that

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \left(1 + \frac{(-1)^n}{\sqrt{n}} \right) = 1.$$

However, while $\sum c_n$ converges by the Alternating Series Test, a quick look at the partial sums

$$\sum_{n=1}^{k} b_n = \sum_{n=1}^{k} \frac{(-1)^n}{\sqrt{n}} + \sum_{n=1}^{k} \frac{1}{n}$$

reveals that $\sum b_n$ diverges.

So this is not the route to establishing the equivalence. However, another tool comes to the rescue [**3**, p. 314]:

Abel's Test. *If the series $\sum c_n$ converges and if the sequence $\{b_n\}$ is monotone and bounded, then the series $\sum b_n c_n$ converges.*

If the series $\sum a_n/n$ converges, then setting $b_n = n/(n+1)$, we deduce that $\sum a_n/(n+1)$ converges. We can obtain the converse using $b_n = (n+1)/n$.

References

[1] H. Anton, *Calculus*, 5th ed. Wiley, 1995.

[2] C. Feist and R. Naimi, Almost alternating harmonic series. *College Math. J.* 35 (2004) 183–191.

[3] K. Knopp, *Theory and Application of Infinite Series*, 2nd ed. Dover, 1990.

4.11 Product of sums equals sum of products

Dan Kalman observes that

$$\sum_{k=0}^{\infty}\left(\frac{1}{7}\right)^k \cdot \sum_{k=0}^{\infty}\left(\frac{-1}{5}\right)^k = \sum_{k=0}^{\infty}\left(\frac{1}{7}\cdot\frac{-1}{5}\right)^k = \frac{35}{36}$$

and that

$$\sum_{k=0}^{\infty}\left(\frac{4}{5}\right)^k \cdot \sum_{k=0}^{\infty}\frac{(-5\ln 5)^k}{k!} = \sum_{k=0}^{\infty}\left(\frac{4^k}{5}\cdot\frac{(-5\ln 5)^k}{k!}\right).$$

Of course, this does not mean that $\sum a_k \cdot \sum b_k = \sum a_k b_k$ in general. But suppose that we are given a convergent series $\sum a_k$. Can we always find a convergent series $\sum b_k$ so that $\sum a_k \cdot \sum b_k = \sum a_k b_k$?

4.12 The harmonic series converges

Hongwei Chen provides the following argument to show that the harmonic series $H = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is convergent in two steps. This seems a very nice example for your students to troubleshoot. For each k with $0 \le k \le 9$, let $H(k)$ denote the sum of the series in which none of the denominators contain the digit k.

For example, when $k = 0$, there are 9 denominators between 1 and 9, inclusive, 9^2 denominators between 10 and 99, inclusive, 9^3 denominators between 100 and 999, inclusive, and so on. Then

$$H(0) \le 9 + 9^2 \times \frac{1}{10} + 9^3 \times \frac{1}{10^2} + \cdots = 90.$$

Similarly, we have that $H(k) \le 90$ for $1 \le k \le 9$.

Secondly, we collect all $H(k)$ $(0 \le k \le 9)$ and get

$$H = H(0) + H(1) + \cdots + H(9) \le 900,$$

so that the series converges. □

4.13 A lopsided interval of convergence

Upsilon: I need to find the Taylor Expansion about $x = 0$ of the function $2(2 + x - x^2)^{-1}$. However, when I start calculating derivatives of this function, the work just gets too painful to contemplate.

Omicron: Well, if you let $y = \frac{1}{2}(x - x^2)$, then you can write

$$\begin{aligned}\frac{2}{2 + x - x^2} = \frac{1}{1 + y} &= 1 - y + y^2 - y^3 + y^4 - \cdots \\ &= 1 - \frac{1}{2}(x - x^2) + \frac{1}{4}(x - x^2)^2 - \frac{1}{8}(x - x^2)^3 + \cdots \\ &= 1 - \frac{1}{2}x + \frac{3}{4}x^2 - \cdots .\end{aligned}$$

Upsilon: But when will it converge?

Omicron: Well, you need to have $|y| < 1$. Since $x - x^2 = x(1 - x)$ never exceeds $1/4$, all you need is to ensure that $x - x^2 > -2$, or that $-1 < x < 2$.

Upsilon: But I thought that the interval of convergence was supposed to be symmetrical about the point of expansion, in this case something of the form $\{x : -R < x < R\}$.

CHAPTER 5

Differential Calculus

5.1 Infectious continuity

Theorem *If the real-valued function f defined on* $\mathbf{R}$ *is continuous at a, then there exists an interval I around a on which f is continuous.*

Proof f is continuous at a. Therefore, for all $\epsilon > 0$, there exists $\delta > 0$ such that, $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon/2$. Choose b in the interval $I = (a - \delta/4, a + \delta/4)$. Clearly $|b - a| < \delta$, and so, for any x in I,

$$\begin{aligned} |f(x) - f(b)| &= |f(x) - f(a) + f(a) - f(b)| \\ &\leq |f(x) - f(a)| + |f(a) - f(b)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

It follows that f is continuous at b. Since b was arbitrary in I, f is continuous on I. □

Contributed by Russ Euler and Jawad Sadek.

Comment One way to get a handle on the situation is to try it on the well-known pathological example $f(x) = 0$ when $x = 0$ or x is irrational, and $x = 1/q$ when $x = p/q$ is a rational number written in lowest terms with $q > 0$. Since this is continuous at 0, but not at any nonzero rational, it serves as a counterexample. If $u > 0$, a small enough positive value of ϵ will require a $\delta < |u|$ to satisfy the definition of continuity at 0, and such a value of δ will not serve at u.

5.2 A simple way to differentiate a quotient

Anand Kumar obtained the following solution from one of his first year calculus students.

Problem Suppose that $f(x) = -(1-(3/x))^{-3}$ and $g(x) = -x^3$. Determine the derivative of $f(x)/g(x)$.

Solution We have that

$$f'(x) = -\left(1-\frac{3}{x}\right)^{-4} \cdot \frac{9}{x^2} = -9x^2(x-3)^{-4} \quad \text{and} \quad g'(x) = -3x^2.$$

Hence

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)}{g'(x)} = 3(x-3)^{-4}. \qquad \square$$

This process gives the correct answer, as indeed it does for $(f(x), g(x)) = ((1-(n/x))^{-n}, x^n)$, $(e^{-x/2}, e^{-x})$ and $(1+(1/x), 1/x)$.

5.3 $x = \sin x$

Omicron: I have to solve the functional equation

$$f(x) - f(y) = (x-y)\cos(x+y)$$

with the additional conditions that $f(0) = 0$ and $f(x)$ is continuous.

Upsilon: That is easy. Divide the equation by $x-y$ and let y tend to x. This leads to $f'(x) = \cos 2x$, whereupon $f(x) = \frac{1}{2}\sin 2x = \sin x \cos x$.

Omicron: An even easier solution is to just set $y = 0$. This leads to $f(x) = x\cos x$.

Upsilon: So then $x = \sin x$.

Omicron: Of course not. Our workings are predicated on the assumption that there *is* a solution to the equation. We have used only a piece of the information embodied in the equation. It may be that we can draw from the equation a set of properties that are inconsistent.

Upsilon: Let us experiment and exploit the fact that $\cos\frac{\pi}{2} = \cos\frac{3\pi}{2} = 0$. If $(x, y) = (\pi, -\frac{\pi}{2})$, the equation yields that $f(\pi) = f(-\frac{\pi}{2})$. If $(x, y) = (\pi, \frac{\pi}{2})$, then $f(\pi) = f(\frac{\pi}{2})$. Now let $(x, y) = (\frac{\pi}{2}, -\frac{\pi}{2})$. Then $f(\frac{\pi}{2}) - f(-\frac{\pi}{2}) = \pi$. These three properties of f are inconsistent. So it turns out that there is no solution to the equation.

Omicron: Is there a function g so that the functional equation

$$f(x) - f(y) = (x - y)g(x + y)$$

is solvable with $f(0) = 0$ and $f(x)$ continuous?

Upsilon: Well, setting $y = 0$ leads to $f(x) = xg(x)$ so that the equation becomes

$$(x + y)(f(x) - f(y)) = (x - y)f(x + y).$$

Dividing by $(x + y)(x - y)$ and letting y tend to x yields that

$$f'(x) = \frac{f(2x)}{2x}$$

whenever $x \neq 0$.

Omicron: It is checked that the space of solutions is a linear space that contains $f(x) = x$ and $f(x) = x^2$, so that

$$(f(x), g(x)) = (ax^2 + bx, ax + b)$$

works. Are there any other solutions?

Upsilon: We have that $f(x) - f(1 - x) = (2x - 1)f(1)$ which when differentiated and manipulated gives

$$(1 - x)f(2x) + xf(2 - 2x) = 4x(1 - x)f(1).$$

Going back to the original equation with (x, y) replaced by $(2x, 2 - 2x)$, we get that

$$f(2x) - f(2 - 2x) = (2x - 1)f(2).$$

The two equations lead to

$$f(2x) = 4x(1 - x)f(1) + x(2x - 1)f(2)$$

whenever $x \neq 0, 1$. We can determine $f(x)$ from this and show that $f(x)$ must have the form $ax^2 + bx$ for constants a and b.

5.4 A strong differentiability condition

Anyone who has written a textbook will understand the power of gremlins, who will manage to slip in mistakes that will elude the most eagle eye. This may help the reader become more tolerant of some flubs that do not seem at all on first glance to be at all out of place; perhaps most readers will never

even notice them. But Sarah V. Cook did pick up the following from the popular textbook, *Calculus with analytic geometry*, Sixth edition, by R.E. Larson, R.P. Hostetler & B.H. Edwards, published by Houghton Mifflin in Boston. On page 106, we find Theorem 2.5:

> The derivative of the sum (or difference) of two differentiable functions is differentiable and is the sum (or difference) of their derivatives.

The mindset of the authors was sufficiently robust that the corresponding error persisted for the product and quotient of differentiable functions (Theorems 2.7 and 2.8 on pages 114 and 116, respectively).

5.5 Functions with the same derivative

David M. Bloom observes that the derivatives of both $\arctan x$ and $\arctan((1+x)/(1-x))$ are equal to $(1+x^2)^{-1}$. One should be able to deduce from this that

$$\arctan\left(\frac{1+x}{1-x}\right) = \arctan x + C$$

for some constant C. Taking $x = 0$ allows us to determine the value $C = \pi/4$. But wait a minute. If we let $x \to \infty$, we find that $\arctan x \to \pi/2$ while $\arctan((1+x)/(1-x)) \to -\pi/4$. This must surely require that $C = -3\pi/4$. Thus, the constant of integration is not so constant after all.

This is quite a nice example to give students to sort out. The relation between the two arctangents is reflected by the identity

$$\tan\left(\theta - \frac{3\pi}{4}\right) = \tan\left(\theta + \frac{\pi}{4}\right) = \frac{1+\tan\theta}{1-\tan\theta}.$$

When we invert the tangent function, we have to take account of its domain of definition, which will then be the range of the arctangent function.

When $x \in (-\infty, 1)$, then $\arctan x \in (-\pi/2, \pi/4)$ and

$$\frac{1+x}{1-x} = \frac{2}{1-x} - 1$$

lies in the interval $(-1, +\infty)$, putting its arctangent in the interval $(-\pi/4, \pi/2)$. Thus

$$\arctan\left(\frac{1+x}{1-x}\right) = \arctan x + \frac{\pi}{4}$$

for $x < 1$. In a similar way, we can argue that

$$\arctan\left(\frac{1+x}{1-x}\right) = \arctan x - \frac{3\pi}{4}$$

for $x > 1$.

It can be checked that $\frac{1}{2}\arccos(1-x^2)(1+x^2)^{-1}$ also has the derivative $(1+x^2)^{-1}$, but only when $x > 0$. For $x < 0$, its derivative is $-(1+x^2)^{-1}$. This provides another example where the range of the validity of the constant difference of two antiderivatives needs careful analysis.

5.6 Tangent howlers

Carl Libis gives two nonstandard methods used by two of his students to determine the equation of the tangent line to the graph of $g(x) = (x+2)^2$ at the point determined by $x_0 = -1$:

Student 1 wrote that

$$y = g'(x) - g(x_0) = 2(x+2) - g(-1) = 2x + 4 - 1 = 2x + 3,$$

while Student 2 obtained

$$\begin{aligned} y &= \frac{g(x+x_0) - g(x)}{x_0} \\ &= \frac{g(x-1) - g(x)}{-1} \\ &= \frac{(x+1)^2 - (x+2)^2}{-1} \\ &= \frac{x^2 + 2x + 1 - x^2 - 4x - 4}{-1} \\ &= \frac{-2x - 3}{-1} \\ &= 2x + 3. \qquad \square \end{aligned}$$

5.7 The function $y = x^{6/7}$ has a node at the origin

Robert J. MacG. Dawson, following the tradition of many modern high school texts, discovers using the graphing calculator that the graph of the equation $y = x^{6/7}$ has a node when $x = 0$. Each of the six images in Figure 1, at scales from 10×10 to $10^{-5} \times 10^{-5}$, shows clearly that the singularity is a node rather than a cusp.

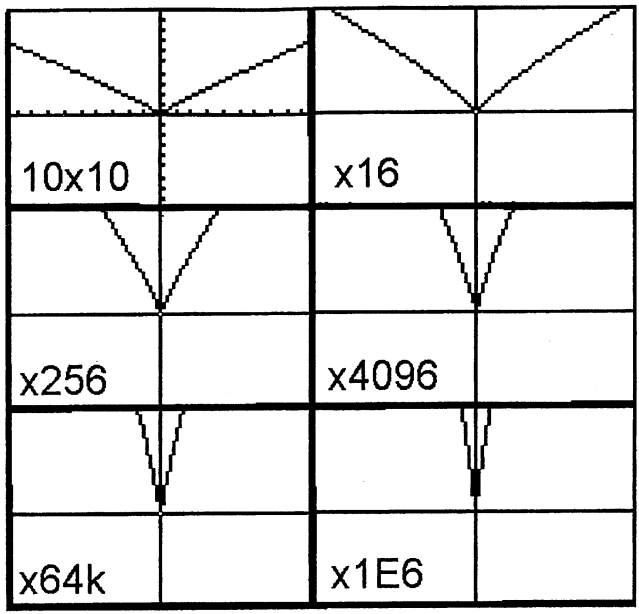


Figure 1.

At magnifications over 1,000,000×, the scale chosen by the calculator makes the nature of the singularity hard to determine. Using *ZoomFit*, however, we can obtain the image in Figure 2, which makes it clear, even at this scale, that the singularity is a node.

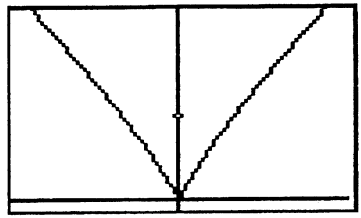


Figure 2.

Dawson notes that images, identical to those generated by his *TI-83 Plus*, were generated by Texas Instruments' excellent *TI-83 Plus* simulator, downloadable from `education.ti.com/us/resources/developer/83/hilight/hilight.html` to which he added the labels afterwards.

5.8 A standard box problem

Dale R. Buske put on a recent calculus examination the following standard problem:

Problem A crate open at the top has vertical sides, a square bottom, and a volume of 4 cubic meters. If the crate is to be constructed so as to have minimal surface area, find its dimensions.

One student started with this formula for the surface area of the crate: $SA = 4x + 4y$, where x was the length of one side of the base and y was the height of the crate. (After all, there are four line segments of length x on the bottom of the crate, four line segments of length y on the sides, and the four line segments on top do not count since the crate has an open top.) The student then correctly went on to use the volume constraint $4 = x^2y$ to find $y = 4/x^2$ and arrive at the formula $SA = 4x + 16/x^2$. Taking the derivative and applying the condition for a maximum leads to the correct answer: the crate should have a base with dimensions 2 meters by 2 meters and a height of 1 meter.

5.9 A gradation of problems

Consider the following optimization problem:

Problem A rancher wishes to fence 200 square yards in a rectangular pasture. One side of the pasture will border a highway, so that the rancher wants *half* of all the money spent on constructing the fence to go toward building that portion of the fence. What dimensions of the pasture will minimize the cost of constructing the fence?

Answer The side bordering the highway should be 20 yards long.

Consider the same problem with a slightly more vain rancher:

Problem A rancher wishes to fence 200 square yards in a rectangular pasture. One side of the pasture will border a highway, so that the rancher wants *two-thirds* of all the money spent on constructing the fence to go toward building that portion of the fence. What dimensions of the pasture will minimize the cost of constructing the fence?

Answer The side bordering the highway should be 20 yards long.

In fact, the answer is the same for any fraction of the cost to be borne by the part bordering the highway. What is going on here?

Thanks to Karl Havlak for this item.

Dan Curtin looks at the problem in this way. Let x be the length of the boundary between field and road. The conditions of the problem require that there be in effect two unit costs for the fencing, according as to whether it is along the road or not. These two unit costs do not maintain a constant ratio with the change in length abutting the road. For a narrow field with the long side abutting the road, the two unit costs will be close, while for a narrow field with the long side perpendicular to the road, the unit cost for the roadside portion will be relatively high. If we assume that the unit cost for the portion away from the road is constant, then the road cost is proportional to $(400/x) + x$, the total length of the three sides not abutting the highway and responsible for half the cost, and we get the answer suggested.

5.10 The area under a tangent

The following question is from an examination at the University of Toronto in Ontario, Canada. The solution was supplied by several students.

Problem Let $f(x) = 2 - x^{1/3}$ for $0 \le x \le 8$.

(a) Suppose that $0 \le u \le 8$. Show that the equation of the tangent to the curve of equation $y = f(x)$ at the point $(u, f(u))$ has the equation

$$y = -\frac{1}{3}u^{-2/3}x + \left(2 - \frac{2}{3}u^{1/3}\right).$$

(b) Let T_u be the triangle bounded by the tangent found in (a) and the two coordinate axes. Determine the value of $u \in [0, 8]$ for which the triangle has maximum area.

Solution to (b) The area of T_u is equal to

$$\frac{1}{2}xy = \frac{1}{2}u(2 - u^{1/3}) = u - \frac{1}{2}u^{4/3}.$$

The derivative $1 - (2u^{1/3}/3)$ is positive for $0 \le u \le 27/8$ and negative for $27/8 \le u \le 8$. Hence, the triangle has its maximum area when $u = 27/8$. □

Comment Surprisingly, the answer turns out to be correct. Is this a fluke? Consider a general curve of equation $y = f(x)$ for $0 \le x \le a$. Suppose that $f(0) > 0$, $f(a) = 0$ and $f'(x) < 0$ on $(0, a)$. The tangent through $(u, f(u))$ has equation $y = f'(u)(x-u) + f(u)$ and its intercepts are $(0, f(u) - uf'(u))$ and $((uf'(u) - f(u))/f'(u), 0)$. Suppose that the tangent cuts the positive quadrant; we see that its area $A(u)$ satisfies

$$2A(u) = -\frac{[f(u) - uf'(u)]^2}{f'(u)}$$

whence

$$2A'(u) = \frac{f''(u)[f(u) - uf'(u)][uf'(u) + f(u)]}{f'(u)^2}.$$

The significant factor in $A'(u)$ for determining the optimum is $uf'(u) + f(u)$, and this is indeed the derivative of $uf(u)$ which gives the answer in the above "solution."

The example $f(x) = 1-x$ with $A(u) \equiv \frac{1}{2}$ and $uf(u) = u(1-u)$ shows that $uf(u)$ is not simply a constant multiple of the area.

5.11 A minimization problem

Problem Let a, b, u, v be real numbers for which $av - bu = 1$. Prove that

$$a^2 + u^2 + b^2 + v^2 + au + bv \ge \sqrt{3}$$

and that the inequality is sharp.

Solution (by a student). Let $x = a^2 + b^2$ and $y = u^2 + v^2$, so that (by the constraint) $xy \ne 0$ and

$$(au + bv)^2 = (a^2 + b^2)(u^2 + v^2) - (av - bu)^2 = xy - 1.$$

Hence $xy \ge 1$. If $au + bv \ge 0$, then

$$\begin{aligned} a^2 + u^2 + b^2 + v^2 + au + bv &= x + y + \sqrt{xy - 1} \ge 2\sqrt{xy} + \sqrt{xy - 1} \\ &\ge 2 \\ &> \sqrt{3}. \end{aligned}$$

Suppose $au + bv < 0$. Then

$$a^2 + u^2 + b^2 + v^2 + au + bv = x + y - \sqrt{xy - 1}.$$

Call the right side $f(x, y)$; we wish to minimize this function. The equation $0 = \partial f/\partial x$ leads to $2 = y(xy - 1)^{-1/2}$ or $4xy - 4 = y^2$. Thus $x = (y^2 + 4)/(4y)$.

We substitute this into $f(x, y)$ to obtain

$$\begin{aligned} g(y) &\equiv \frac{y^2+4}{4y} + y - \sqrt{\frac{y^2+4}{4} - 1} \\ &= \frac{y^2+4}{4y} + y - \frac{y}{2} = \frac{3y^2+4}{4y} \\ &= \frac{3y}{4} + \frac{1}{y}. \end{aligned}$$

To minimize $g(y)$, we solve $0 = g'(y) = \frac{3}{4} - y^{-2}$ to get $y = 2/\sqrt{3}$. (The second derivative is positive at this value.) Thus $f(x, y)$ achieves its minimum value of $\sqrt{3}$ when $(x, y) = (2/\sqrt{3}, 2/\sqrt{3})$.

To achieve the value $\sqrt{3}$ in the left member of the inequality, we need to satisfy $a^2 + b^2 = u^2 + v^2 = 2/\sqrt{3}$ and $av - bu = 1$. The quadruple

$$(a, b, u, v) = (3^{1/4} \cdot 2^{-1/2}, -2^{-1/2} \cdot 3^{-1/4}, 0, 3^{-1/4} \cdot 2^{1/2})$$

will do. □

Comment Even without the constraint to complicate matters, we still have the issue as to the following method for optimizing a two-variable function $f(x, y)$: (1) Solve the equation $\partial f/\partial x = 0$ and obtain one variable in terms of the other; (2) make the substitution into $f(x, y)$; (3) minimize $f(x, y)$ as a function of one variable.

Let us suppose that we are looking for a minimum of the function $f(x, y)$. Ignoring boundary possibilities, the minimizing point lies on the curve C defined by $\frac{\partial f}{\partial y}(x, y) = 0$. In general, such a curve can be parameterized by a single variable t (which might be x or y). The function $f(x, y)$, restricted to C can be written in terms of t and its minimum on C determined.

5.12 A solution to savor

Problem A brine tank containing 3 kg of salt per liter runs into a tank filled with 400 L of water containing 20 kg of salt. The brine enters at 10 L/min and flows out at the same rate. Find the mass of salt in the tank after 10 minutes.

Solution (by a student) Let $A(t)$ denote the mass of salt in the tank in kg after t minutes, The input rate of salt is 30 kg/min and the output rate is $10 \times (t/400)$ kg/min. Hence

$$\frac{dA}{dt} = 30 - \frac{t}{40},$$

with the initial condition $A(0) = 20$. Integrating yields

$$A(t) = 30t + \frac{(-t)^2}{80} + C,$$

so applying the initial condition gives $C = A(0) = 20$ and thus

$$A(t) = 30t + \frac{(-t)^2}{80} - 20. \quad \text{(sic!)}$$

Hence, at time $t = 10$, we find that $A(10) \approx 281$ kg. □.

The solution to this textbook problem was provided by a student of Dale R. Buske. It turns out that the answer is almost correct. The differential equation should be $A' = 30 - A/40$ with solution $A(t) = 1200 - 1180e^{-t/40}$; $A(10) = 1200 - 1180e^{-1/4} \approx 281$ kg.

5.13 Falling ball

Karl Havlak presents the following student solution to a standard problem, where several errors lead to the right solution. He proposes the task of explaining to the solver why full credit was not given when the time taken to strike the ground and the height of the building were correctly determined.

Problem A ball is dropped from the top of a building and hits the ground with a speed of 128 feet per second. Approximate the height of the building.

Solution Using the formula $s(t) = -(g/2)t^2 + v_0 t + s_0$, where $g = 32$ is the gravitational constant, v_0 the initial velocity and s_0 the initial height of the ball, we obtain

$$s(t) = -16t^2 + 128t.$$

When the ball hits the ground, the velocity is 0 for an instant, so we have

$$s'(t) = -32t + 128 = 0 \Longrightarrow t = 4.$$

Since the ball fell for 4 seconds, we obtain the height of the building to be

$$s(4) = -16(4^2) + 128(4) = 256 \qquad \square.$$

5.14 A fallacy that wasn't

Differential equations books generally mention Euler's theorem on homogeneous functions in a section discussing substitution methods for solving first order differential equations. A homogeneous function $F(x, y)$ of degree n is defined by the property that

$$F(tx, ty) = t^n F(x, y). \qquad (*)$$

Euler's theorem provides that, for such a function,

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = nF.$$

The standard proof is to first differentiate the equation $(*)$ with respect to t, getting

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = nt^{n-1}F,$$

and then to set $t = 1$. However, Bill Gerson reflects that one could get different results by setting t equal to other values.

For example, if we take $t = 2$, then we obtain the result

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = n2^{n-1}F.$$

What is going on here?

Not very much, avers Gerson. The seeming difficulty arises from the misleading notation rather than from a logical flaw. Go back to equation $(*)$ and consider more carefully the effect of the chain rule when the left side is differentiated with respect to t. The result is the sum of two terms, the first being the first partial of F evaluated at the point (tx, ty) multiplied by the derivative of tx with respect to t and the second term similarly structured. Thus, we find that

$$x\frac{\partial F(tx, ty)}{\partial x} + y\frac{\partial F(tx, ty)}{\partial y} = nt^{n-1}F(x, y).$$

Now set $t = 2$ and compare with the $t = 1$ case.

Looking at a simple example makes the situation plain. Let $F(x, y) = x^2 + xy + y^2$, so that $F(tx, ty) = (tx)^2 + (tx)(ty) + (ty)^2$. Then

$$\frac{\partial F(tx, ty)}{\partial t} = [2(tx)x + (ty)x] + [(tx)y + 2(ty)y] = 2t(x^2 + xy + y^2).$$

Evaluating this at $t = 2$ yields $4(x^2 + xy + y^2)$, which is equal to $n2^{n-1}F(x, y)$ for $n = 2$.

CHAPTER 6

Integral Calculus

6.1 Integration discrepancies?

Having students perform integration by hand is one way to help them become familiar with the properties of standard functions and to develop judgment as to the best way to proceed. However, different approaches can lead to results that might appear at variance with each other. It is a nice exercise for students to reconcile them. Here are some examples submitted by readers of the *College Mathematics Journal* over the years.

Example 1 Integrate $\int \frac{x}{(x-1)^2} dy$.

Solution 1 An integration by parts with $u = x$ and $dv = (x-1)^{-2}dx$ yields the answer

$$\int \frac{x}{(x-1)^2}dx = \frac{-x}{x-1} + \int \frac{1}{x-1}dx = \frac{-x}{x-1} + \ln(x-1) + C.$$

Solution 2 The substitution $x = 1 + u$ leads to

$$\begin{aligned}\int \frac{x}{(x-1)^2}dx &= \int \frac{u+1}{u^2}du \\ &= \int \frac{1}{u}du + \int \frac{1}{u^2}du \\ &= \ln u - \frac{1}{u} + C = \ln(x-1) - \frac{1}{x-1} + C.\end{aligned}$$

Submitted by Ken Taylor.

Example 2 Integrate

$$\int \frac{1}{\sqrt{x-x^2}}dx.$$

Solution 1 Completing the square under the radical in the denominator yields

$$\int \frac{1}{\sqrt{x-x^2}}dx = \int \frac{2}{\sqrt{1-(2x-1)^2}}dx = \arcsin(2x-1)+C. \qquad \square$$

Solution 2 Factoring the denominator as $\sqrt{1-x}\sqrt{x}$ and substituting $u = \sqrt{x}$ yields

$$\int \frac{1}{\sqrt{x-x^2}}dx = 2\int \frac{1}{\sqrt{1-u^2}}du = 2\arcsin(\sqrt{x}) + C. \qquad \square$$

Comment To sort this out, let $\alpha = \arcsin(2x-1)$ and $\beta = \arcsin\sqrt{x}$. Then $0 \le x \le 1$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, and α and β are increasing functions of x. Sampling the values $x = 0, \frac{1}{2}, 1$ leads us to believe that $2\beta = \frac{\pi}{2} + \alpha$. Indeed,

$$\begin{aligned}\sin 2\beta &= 2\sin\beta\cos\beta = 2\sqrt{x(1-x)} \\ &= \sqrt{1-(2x-1)^2)} = \cos\alpha = \sin\left(\frac{\pi}{2}+\alpha\right).\end{aligned}$$

Therefore $2\beta = \frac{\pi}{2} + \alpha$, or $2\arcsin\sqrt{x} = \frac{\pi}{2} + \arcsin(2x-1)$.

Roger B. Nelsen found these answers to an integration when he was grading an examination.

Example 3 Ollie Nanyes draws attention to the fact that, depending on the substitution, one obtains variously

$$\int \cosh x \sinh x = \frac{1}{2}\cosh^2 x + C$$

and

$$\int \cosh x \sinh x = \frac{1}{2}\sinh^2 x + C.$$

Example 4 However, the issues are more subtle in the following, also from Ollie Nanyes:

$$\begin{aligned}\int \csc x\,dx &= \int \csc x \frac{\cot x + \csc x}{\cot x + \csc x}dx \\ &= \int \frac{\csc x\cot x + \csc^2 x}{\cot x + \csc x}dx \\ &= -\ln|\csc x + \cot x| + C\end{aligned}$$

$((k-1)\pi < x < k\pi)$, k an integer. We also have

$$\begin{aligned}\int \csc x\,dx &= \int \csc x \frac{\csc x - \cot x}{\csc x - \cot x}dx \\ &= \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x}dx \\ &= \ln|\csc x - \cot x| + C\end{aligned}$$

$((k-1)\pi < x < k\pi)$, k an integer. Note that the absolute value signs are essential to the solution.

Example 5 David M. Bloom reports that he has shown his class alternative indefinite integrals of the secant function. A table of values gives $\int \sec x = \ln|\sec x + \tan x| + C$, while writing the integrand as

$$\sec x = \frac{\cos x}{1 - \sin^2 x}$$

and making the substitution $u = \sin x$ leads, by a standard partial fraction evolution, to the answer

$$\frac{1}{2}(\ln|1+u| - \ln|1-u|) = \frac{1}{2}(\ln|1+\sin x| - \ln|1-\sin x|) + C.$$

Here the constants C are actually the same in both expressions, as can be verified using a little trigonometry.

Example 6 Karl Havlak looks at the integral

$$\int \frac{x}{\sqrt{x+1}}dx.$$

Using the parts $u = x$ and $dv = (x+1)^{-1/2}dx$ leads to the indefinite integral

$$2x(x+1)^{1/2} - \frac{4}{3}(x+1)^{3/2} + C,$$

while the substitution $t = x + 1$ yields

$$\frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C.$$

Example 7 Paul H. Schuette made use of symbolic software to compute the indefinite integral

$$\int \frac{dx}{x\sqrt{1-x^2}}.$$

Maple 7 and *Maple 9.5* both gave

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\text{arctanh}\left(\frac{1}{\sqrt{1-x^2}}\right).$$

The integrand is defined as a real function only for $0 < |x| < 1$. However, on this domain, the range of $1/\sqrt{1-x^2}$ is $(1, \infty)$, while the domain of the real function arctanh$x$ is $(-1, 1)$. Thus, the antiderivative has a null domain.

To gain some insight into the situation, Schuette suggests examining the function

$$g(x) = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|$$

defined whenever $|x| \neq 1$. For $-1 < x < 1$, $g(x) = \text{arctanh}x$. However, when $|x| > 1$ and arctanh$x$ is undefined, $g(x) \neq \text{arctanh}x$. Thus $g(1/\sqrt{1-x^2}) \neq \text{arctanh}(1/\sqrt{1-x^2})$. Using the substitution $x = \sin\theta$, it can be shown that

$$\int \frac{dx}{x\sqrt{1-x^2}} = -g\left(\frac{1}{\sqrt{1-x^2}}\right) + C, \qquad (*)$$

for $0 < |x| < 1$.

Maple is not the only computer algebra system that exhibits this sort of domain problem. Version 6 of *Derive* reports that

$$\int \frac{dx}{x\sqrt{1-x^2}} = \ln\left(\frac{\sqrt{1-x^2}-1}{x}\right),$$

which agrees with $(*)$, but only when $-1 < x < 0$. On the other hand, *Mathematica* reports that

$$\int \frac{dx}{x\sqrt{1-x^2}} = \ln x - \ln(1+\sqrt{1-x^2}),$$

which agrees with $(*)$ only when $0 < x < 1$.

David W. Cantrell, objected that a casual reader would take away the erroneous impression that *Derive*, *Maple*, and *Mathematica* gave results that were incorrect. He points out that all the antiderivatives are correct.

He compares the situation with that of integrating $1/x$; all CAS systems he is aware of give $\log x$, while textbooks give $\log|x|$, plus an arbitrary constant of integration. Why the discrepancy? "CAS are normally designed to give results applicable in the complex, not just real, realm." While $\log|x|$ is fine for elementary calculus, $\log x$ is not wrong. When $x < 0$, the logarithms of x and $|x|$ differ by a constant, namely the imaginary πi.

Likewise, *all* the antiderivatives of $1/x\sqrt{1-x^2}$ are correct. "In particular, for $0 < |x| < 1$, they all have the same real part. Their only differences, for $-1 < x < 0$ or for $0 < x < 1$, are merely *constants*, albeit imaginary ones.

6.2 Making an integrand completely imaginary

John K. Osoinach spends quite a bit of time in second semester calculus on the mechanics of integration, such as finding "good" substitutions, and rewriting integrands into a more convenient form. Usually, the mistakes that students make are obvious and easily corrected, but recently a very bright student challenged him to find the mistake in the following solution. This led to a useful conversation about how issues from algebra are still relevant, even if they have not arisen for quite some time.

Exercise Find the integral

$$\int \frac{1}{\sqrt{-x^2-4x}}dx.$$

Solution Factoring out an x gives

$$\int \frac{1}{\sqrt{x(-x-4)}}\,dx.$$

Separate the denominator into two radicals:

$$\int \frac{1}{\sqrt{x}\sqrt{-x-4}}\,dx.$$

Make the substitution $u = \sqrt{x}$, so that $du = (1/2\sqrt{x})dx$, which yields

$$\int \frac{2}{\sqrt{-u^2-4}}\,du.$$

Since the quantity inside the radical is always negative, the function cannot be integrated. □

Note that in the original integrand, the quantity inside the radical is positive for $-4 < x < 0$. Accordingly, a more reasonable substitution would be $u = \sqrt{-x}$, so that $x = -u^2$, $dx = -2u\,du$ and the integral is

$$\int \frac{1}{\sqrt{-x}\sqrt{4+x}}dx = \int \frac{-2du}{\sqrt{4-u^2}} = -2\arcsin\frac{u}{2} = -2\arcsin\frac{\sqrt{-x}}{2} + c.$$

6.3 A funky change of variables

Allen J. Schwenk reports that one of his students was inspired to try the change of variables $u = x^2$ in calculating the definite integral $\int_{-1}^{1} 2x^2 dx$. Since $du = 2x\,dx$, the integral becomes $\int_1^1 \sqrt{u}\,du$. Since the lower and upper limits are identical, the latter integral vanishes. What is the problem?

Apparently, the substitution should work as $u' = 2x$ is continuous on $[-1, 1]$. However, when $x < 0$, $x = -\sqrt{u}$. Splitting the interval of integration into two parts, $[-1, 0]$ and $[0, 1]$ and substituting $x = -\sqrt{u}$ on the first and $x = \sqrt{u}$ on the second will lead to a correct result.

6.4 An integration by parts

Stephen Scarborough found that a student on a test decided to find $\int(x + x^2)dx$ by integrating by parts, but nothing seemed to work out for him. Scarborough wondered whether the method had any hope of success, and presents the following for our amusement.

Using $u = x$ in the standard formula, we have that

$$\begin{aligned}
&\int (x + x^2)dx \\
&= x\left(x + \frac{1}{2}x^2\right) - \int \left(x + \frac{1}{2}x^2\right)dx \\
&= x\left(x + \frac{1}{2}x^2\right) - x\left(x + \frac{1}{4}x^2\right) + \int \left(x + \frac{1}{4}x^2\right)dx = \cdots \\
&= x\left(x + \frac{1}{2}x^2\right) - x\left(x + \frac{1}{4}x^2\right) + x\left(x + \frac{1}{8}x^2\right) - x\left(x + \frac{1}{16}x^2\right) + \cdots \\
&= (1 - 1 + 1 - 1 + \cdots)x^2 + \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots\right)x^3 \\
&= \frac{1}{2}x^2 + \frac{1}{3}x^3,
\end{aligned}$$

where we accept the standard determination $1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$.

Thus, $\int(x + x^2)dx = \frac{1}{2}x^2 + \frac{1}{3}x^3 + C$. □

6.5 Integrating without finding the antiderivative

Anand Kumar contributes another definite integral that can be correctly calculated without finding an antiderivative for the integrand:

$$\int_1^2 (3x^2 + 4x)dx = \Big[3x^2 + 4x\Big]_1^2 = 20 - 7 = 13.$$

Other examples appear in FFF#102 (*CMJ* 27 (1996), 118): $\int_{-1}^{2} x^2 dx$ and $\int_{-1}^{2}(x+1)^2 dx$.

6.6 Differentiating an integral

John A. Quintanilla provides another distributivity mishap by a student, who nonetheless gets the correct result.

Problem If

$$f(x) = \int_2^{\sqrt{x}} \frac{1}{t^2 - 1} dt,$$

calculate $f'(x)$.

Solution

$$\begin{aligned}
f(x) &= \int_2^{\sqrt{x}} \frac{1}{t^2-1} dt \\
&= \int_2^{\sqrt{x}} \left(\frac{1}{t^2} - \frac{1}{1}\right) dt \\
&= \left[-\frac{1}{t} - t\right]_2^{\sqrt{x}} \\
&= \left[-\frac{1}{\sqrt{x}} - \sqrt{x}\right] - \left[-\frac{1}{2} - 2\right] \\
&= -\frac{1}{\sqrt{x}} - \sqrt{x} + \frac{5}{2} \\
f'(x) &= \frac{1}{2x^{3/2}} - \frac{1}{2\sqrt{x}} \\
&= \frac{1}{2x^{3/2} - 2\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}(x-1)}.
\end{aligned}$$

6.7 Several wrongs make a right

Exercise Find the volume of the solid obtained when the region bounded by the curves $x = y^2$ and $x = 2y$ is rotated about the $y-$axis.

Solution (by a student) The answer is

$$\begin{aligned}\pi \int_0^4 \Big((y^2)^2 - (2y)^2\Big)dy &= \pi \left| \frac{y^5}{5} - \frac{4y^3}{3} \right|_0^4 \\ &= \frac{4\pi}{(5)(3)} |y^5 - y^3|_0^4 \\ &= \frac{4\pi}{15} |y^2|_0^4 \\ &= \frac{64\pi}{15}. \qquad \square\end{aligned}$$

The answer is correct. This student solution was submitted by Carl Libis.

6.8 The generality of the trapezoid rule

Problem Evaluate the integral

$$\int_0^1 \frac{x^2}{2x^2 - 2x + 1} dx.$$

Solution (by a student) Using the formula

$$\int_a^b f(x)dx = \frac{1}{2}(f(a) + f(b))$$

for the given integral, we obtain its value $(0 + 1)/2 = 1/2$. $\square$

Comment The answer is not only correct, but the method works under the proper conditions, when the length of the interval is taken into account. Suppose that $f(x)$ is defined on $[a, b]$ and that $f(x) + f(a + b - x)$ is a constant, there, independent of x. The constant value must be $f(a) + f(b)$. Since $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$, we have that

$$\int_a^b f(x)dx = \frac{1}{2} \int_a^b [f(x) + f(a+b-x)]dx = \frac{1}{2}[f(a) + f(b)](b-a).$$

This is the case in the present problem, as the integrand can be written as $x^2/[x^2 + (1 - x)^2]$. This item is from M.A. Khan.

6.9 The integral of a positive function equals 0

Hongwei Chen produces a nice example of a function that is everywhere positive on an interval, yet its integral can be shown to vanish. The reader may wish to compare this with another example discussed many years ago (*CMJ* 20 (1989), 226, 318).

We show that

$$\int_0^1 \frac{x^4+1}{x^6+1}dx = 0.$$

We begin by working out the indefinite integral

$$I = \int \frac{x^4+1}{x^6+1}dx.$$

Since $x^6+1 = (x^2+1)(x^4-x^2+1)$, the corresponding partial fraction decomposition yields that

$$I = \int \frac{2}{3(x^2+1)}dx + \int \frac{x^2+1}{3(x^4-x^2+1)}dx.$$

Since

$$\int \frac{x^2+1}{x^4-x^2+1}dx = \int \frac{1+1/x^2}{x^2-1+1/x^2}dx = \int \frac{1+1/x^2}{(1/x-x)^2+1}dx,$$

the substitution $u = 1/x - x$ gives

$$\begin{aligned}\int \frac{1+1/x^2}{(1/x-x)^2+1}dx &= -\int \frac{du}{1+u^2}\\ &= -\arctan(1/x-x) + C_1\\ &= \arctan\left(\frac{x}{1-x^2}\right) + C_2,\end{aligned}$$

where we have used the fact that

$$\arctan t + \arctan(1/t) = \pm\frac{\pi}{2}.$$

Recalling that

$$\arctan\alpha + \arctan\beta = \arctan\left(\frac{\alpha+\beta}{1-\alpha\beta}\right),$$

we arrive at

$$\begin{aligned}I &= \frac{1}{3}\left(2\arctan x + \arctan\left(\frac{x}{1-x^2}\right)\right) + C\\ &= \frac{1}{3}\arctan\left(\frac{3x(1-x^2)}{x^4-4x^2+1}\right) + C.\end{aligned}$$

Therefore,

$$\int_0^1 \frac{x^4+1}{x^6+1}dx = \frac{1}{3}\arctan\left(\frac{3x(1-x^2)}{x^4-4x^2+1}\right)\Bigg|_0^1 = 0. \qquad \square$$

Comment We have to note that the function $x^4 - 4x^2 + 1$ vanishes at the value $\nu = \sqrt{2-\sqrt{3}}$ in the interval $[0, 1]$. On the interval $[0, \nu]$, the function in large parentheses increases from 0 without bound, and on the interval $(\nu, 1]$, it is unbounded, negative and increases to 0. Thus

$$\begin{aligned}\int_0^1 \frac{x^4+1}{x^6+1} &= \int_0^\nu \frac{x^4+1}{x^6+1}dx + \int_\nu^1 \frac{x^4+1}{x^6+1}dx \\ &= \frac{1}{3}\left[\frac{\pi}{2} - 0\right] + \frac{1}{3}\left[0 - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{3}.\end{aligned}$$

6.10 Consequences of an integral equality

The solution to the problem was supplied by a student.

Problem Suppose that $f(x)$ is a continuous real-valued function defined on the interval $[0, 2]$ for which

$$\int_0^2 f(x)dx = \int_0^2 (f(x))^2 dx.$$

Prove that there exists a number $c \in [0, 2]$ for which either $f(c) = 0$ or $f(c) = 1$.

Solution Clearly $\int_0^2 f(x) \geq 0$. By the extreme value theorem, there exist numbers u and v in $[0, 2]$ for which $f(u) \leq f(x) \leq f(v)$ for $0 \leq x \leq 2$. Hence

$$f(u)\int_0^2 f(x)dx \leq \int_0^2 f(x)^2 dx \leq f(v)\int_0^2 f(x)dx.$$

Since $\int_0^2 f(x)^2 dx = 1 \cdot \int_0^2 f(x)dx$, by the intermediate value theorem, there exists a number $c \in [0, 2]$ for which $f(c) = 1$. $\square$

Comment The solution is correct for a function which is nonnegative at every point of the interval. However, it fails whenever $f(x)$ assumes a negative value. In this case, there are values of x for which $f(u) \leq f(x) < 0$, so that $f(u)f(x) \geq f(x)^2$ and so

$$f(u)\int_N f(x)dx \geq \int_N \big(f(x)\big)^2 dx$$

where N is the subset of $[0, 2]$ upon which $f(x)$ is negative. This puts into question the displayed inequality in the purported solution.

To see that $f(x)$ need not assume the value 1, we construct an example. Observe that when $f(x) = \frac{1}{2}x$, then $\int_0^2 f(x)dx > \int_0^2 (f(x))^2 dx$, while when $f(x) = \frac{1}{2}(x-2)$, then $\int_0^2 f(x)dx < 0 < \int_0^2 (f(x))^2 dx$. This suggests that we can satisfy the condition of the problem with a function of the form $f(x) = \frac{1}{2}(x-\lambda)$ for some value of λ in the interval $(0, 2)$. Any such function will never assume the value 1 on the interval.

We have that

$$\int_0^2 f(x)dx = 1 - \lambda$$

and

$$\int_0^2 \big(f(x)\big)^2 dx = \frac{1}{6}\big[3(1-\lambda)^2 + 1\big].$$

These two expressions are equal when $\lambda = \sqrt{2/3}$.

We can complete the proof by considering the possibility that $f(x)$ takes negative values. Since the integral of $f(x)$ over $[0, 2]$ is nonnegative, $f(x)$ must assume positive values as well. Therefore, by the intermediate value theorem, it must vanish somewhere within the interval.

Alternatively, we can get a more direct argument by considering the function $g(x) = (f(x))^2 - f(x)$. By hypothesis, $g(x)$ is a continuous function for which $\int_0^2 g(x) = 0$. Therefore, either $g(x)$ vanishes identically or it takes both positive and negative values. By the intermediate value theorem, there exists a number c for which $g(c) = 0$.

6.11 A new "proof" of an old result

William Grounds provides the following proof that $1-1+1-1+1-\cdots = \frac{1}{2}$.

$$\begin{aligned}\frac{1}{2} &= \int_0^\infty e^{-2t}dt = \int_0^\infty e^{-t}e^{-t}dt \\ &= \int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n t^n e^{-t}}{n!}dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\int_0^\infty t^n e^{-t}dt \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{n!}n! = \sum_{n=0}^\infty (-1)^n.\end{aligned}$$

6.12 The integral of the derivative of any integrable function vanishes

Larry Glasser has provided some integral manipulations to demonstrate that the integral of the derivative of any integrable function vanishes. Let us begin with some general results for any real integrable function F defined on the real line and an arbitrary real constant a.

$$\int_{-\infty}^{\infty} F\left(x - \frac{a^2}{x}\right)dx = \int_{-\infty}^{\infty} F(x)dx. \tag{1}$$

This is clear when $a = 0$. Suppose that $a \neq 0$. To establish (1), write $u = x - (a^2/x)$, whence

$$x = \frac{u}{2} + \frac{1}{2}(u^2 + 4a^2)^{1/2}$$

covers $x > 0$ for all real u, and

$$x = \frac{u}{2} - \frac{1}{2}(u^2 + 4a^2)^{1/2}$$

covers $x < 0$ for all real u. Then

$$\begin{aligned}\int_{-\infty}^{\infty} F(x - a^2/x)dx &= \int_0^{\infty} F(x - a^2/x)dx + \int_{-\infty}^{0} F(x - a^2/x)dx \\ &= \int_{-\infty}^{\infty} F(u)\left[\frac{1}{2} + \frac{u}{2}(u^2 + 4a^2)^{-1/2}\right]du \\ &\quad + \int_{-\infty}^{\infty} F(u)\left[\frac{1}{2} - \frac{u}{2}(u^2 + 4a^2)^{-1/2}\right]du \\ &= \int_{-\infty}^{\infty} F(u)\,du. \qquad \square\end{aligned}$$

We have, also, that

$$\int_{-\infty}^{\infty} F\left(x - \frac{a^2}{x}\right)\frac{dx}{x} = 0. \tag{2}$$

This is effected through two changes of variable; $u = -x$ and $v = a^2/u$:

$$\begin{aligned}\int_{-\infty}^{0} F\left(x - \frac{a^2}{x}\right)\frac{dx}{x} &= -\int_0^{\infty} F\left(-u + \frac{a^2}{u}\right)\frac{du}{u} \\ &= \int_{\infty}^{0} F\left(-\frac{a^2}{v} + v\right)\frac{dv}{v} \\ &= -\int_0^{\infty} F\left(x - \frac{a^2}{x}\right)\frac{dx}{x},\end{aligned}$$

from which the result follows.

Finally,

$$\int_{-\infty}^{\infty} F\left(x - \frac{a^2}{x}\right)\frac{dx}{x^2} = \frac{1}{a^2}\int_{-\infty}^{\infty} F(x)dx, \tag{3}$$

when $a \neq 0$. To see this, make the change of variable $w = -a^2/x$ and use (1):

$$\begin{aligned}\int_{-\infty}^{\infty} F\left(x - \frac{a^2}{x}\right)\frac{dx}{x^2} &= \frac{1}{a^2}\int_{-\infty}^{\infty} F\left(w - \frac{a^2}{w}\right)dw \\ &= \frac{1}{a^2}\int_{-\infty}^{\infty} F(w)dw. \qquad \square\end{aligned}$$

With these preliminaries, let $f(x) = g'(x)$, where $g(x)$ is an integrable function. Then, applying (2) to g, we have that

$$\phi(a) \equiv \int_{-\infty}^{\infty} g\left(x - \frac{a^2}{x}\right)\frac{dx}{x} = 0$$

for any nonzero constant a. Differentiation with respect to a yields

$$0 = \phi'(a) = -2a\int_{-\infty}^{\infty} f\left(x - \frac{a^2}{x}\right)\frac{dx}{x^2} = -\frac{2}{a}\int_{-\infty}^{\infty} f(x)dx$$

by (3). Thus, $\int_{-\infty}^{\infty} f(x)dx = 0$.

For example, since $(2/\sqrt{\pi})e^{-x^2} = \frac{d}{dx}[\text{erf}(x) - 1]$,

$$\int_{-\infty}^{\infty} e^{-x^2}dx = 0.$$

6.13 A quick way to get arc length

The following problem posed by E.T.H. Wang appeared in *Crux Mathematicorum* 26 (2000), 179; 27 (2001), 282.

> On a recent calculus test, students were asked to compute the arc length of a curve represented by a certain function $f(x)$, for $x = a$ to $x = b$, $a < b$. One of the students, a Mr. Fluke, simply calculated $f'(b) - f'(a)$, and obtained the correct answer. Determine all real functions, $f(x)$, differentiable on some open interval I, such that, for all a, b satisfying $(a, b) \in I$, the arc length of the curve $y = f(x)$, from $x = a$ to $x = b$ is equal to $f'(b) - f'(a)$.

Robert Weinstock points out that we can deduce from the Fundamental Theorem of Calculus and the arc length formula that

$$f''(x) = \sqrt{1 + f'(x)^2}.$$

Since $f''(x)^2 - f'(x)^2 = 1$, we can infer that, for each x, there exists a unique real $u(x)$ for which $f''(x) = \cosh u(x)$ and $f'(x) = \sinh u(x)$. Differentiating the latter yields that $f''(x) = u'(x)\cosh u(x)$, whence $u'(x) = 1$ and $u(x) = x - \alpha$ for some constant α. Therefore, $f'(x) = \sinh(x - \alpha)$, so that $f'(x) = \cosh(x - \alpha) + \beta$, for some constant β. It can be verified that each such function works.

6.14 Spherical volume via cylindrical coordinates

James Swenson taught his students how to change variables in multiple integrals. He assigned his students the task of computing the volume of a sphere of radius r. He "made a big mistake" by not denoting the radius by R. Rather than using spherical coordinates to evaluate the relevant triple integral as anticipated, several students, frightened perhaps by the "ominous Jacobian determinant $\rho^2 \sin\phi$" turned in solutions like this one:

$$\begin{aligned} V &= \int\int\int_B dV = \int_0^{2\pi}\int_0^r\int_{-r}^r r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^r 2r^2\,dr\,d\theta = \int_0^{2\pi}\frac{2}{3}r^3\,d\theta \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

The students have used absolute bounds on each variable as the limits of integration, and have made the symbol r do double duty as a constant and as a variable. In effect, the ball B has been replaced by a cylinder from which a double cone has been removed. The fact that

$$(\pi r^2)(2r) - 2\left[\frac{1}{3}(\pi r^2)(r)\right] = \frac{4}{3}\pi r^3$$

is unfortunate for students and grader alike.

In fact, a derivation of the volume of a sphere is obtainable using Cavalieri's Method, where the comparison solid is such a cylinder with a double cone removed.

6.15 An epidemic of Jacobians

The following was written by Edward Aboufadel of Grand Valley State University, Allendale, MI:

In my differential equations course (winter, 2000), I assigned the following problem dealing with the stability of an equilibrium point of a system of equations that models viral dynamics. The problem was in a *Maple* worksheet, and the students were expected to use *Maple* to analyze the system.

The Problem In the January 28, 2000, issue of *Science* [[1]], an *SEIR* model is described to analyze measles in populations in New York, Baltimore, London and Liverpool. The system of four equations is stated below, with the names of constants changed to better fit their meaning.

In the equations below, $E(t)$ is the percentage exposed, but not yet technically infected, while $S(t)$ is the percentage that is susceptible, but not yet exposed. Instead of $I(t)$, we have $i(t)$, because I is reserved in *Maple* for the square root of -1, and this variable stands for the percentage infected but not recovered. The percentage recovered is $R(t)$. Therefore $S + E + i + R = 1$.

The birth rate per capita of the population is b and the death rate per capita is d. The rate of disease transmission is r.

The latency period, the amount of time between being exposed and being infected, is the reciprocal of l (so the shorter the latency period, the longer l is), and the infection period, the amount of time you are actually sick with measles, is the reciprocal of g.

Here are the equations:

$$\frac{dS(t)}{dt} = b - ri(t)S(t) - dS(t)$$
$$\frac{dE(t)}{dt} = ri(t)S(t) - lE(t) - dE(t)$$
$$\frac{di(t)}{dt} = lE(t) - gi(t) - di(t)$$
$$\frac{dR(t)}{dt} = gi(t) - dR(t).$$

In the *Science* article, they give the following values: $g = 0.2$ per day, $l = 0.125$ per day, and $b = d = 0.02$ per year. (In other words, the infection period is 5 days, the latency period is 8 days, and the birth and death rates are 2% per year. The b term in the first equation comes from b times 100%, and we assume everyone who is born is automatically susceptible.) We will use $r = 700$.

One of the equilibrium points of the system is $(1, 0, 0, 0)$, the situation before a brand-new disease is introduced into the population, and everyone

is susceptible. Determine the other equilibrium point of the system. Compute the Jacobian matrix for the more interesting equilibrium point. Then compute eigenvalues to decide if it is a source, sink, or node. □

The other equilibrium point turns out to be

$$(.0003645714286, .07834133453, .7834133453, .1378807488).$$

Most students then used *Maple* to compute the Jacobian matrix, and its eigenvalues, doing some rounding of calculations in the process, and putting the rows in the same order as in the original equation.

>J:= matrix (4, 4, [-54.83, 0, -.255, 0, 54.83, -.145, .255, 0, 0, .125, -.22, 0, 0, 0, .2, -,02]);

$$J := \begin{bmatrix} -54.83 & 0 & -.255 & 0 \\ 54.83 & -.145 & .255 & 0 \\ 0 & .125 & -.22 & 0 \\ 0 & 0 & .2 & -.02 \end{bmatrix}.$$

>eigenvalues(J);

$$-.02000000000, -.1461536437, -.2182611306, -54.83058523$$

As all of the eigenvalues are negative, the equilibrium point is stable. (This agreed with the results in the *Science* paper.) A few students, however, set up a different Jacobian matrix, as if they were working with an $RSEI$ model rather than a $SEIR$ model:

>K:= matrix (4, 4, [0, 0, .2, -.02, -54,83, 0, -.255, 0, 54.83, -.145, .255, 0, 0, .125, -.22, 0]);

$$K := \begin{bmatrix} 0 & 0 & .2 & -.02 \\ -54.83 & 0 & -.255 & 0 \\ 54.83 & -.145 & .255 & 0 \\ 0 & .125 & -.22 & 0 \end{bmatrix}.$$

This time

>eigenvalues(K);

$$3.530380744, -3.095381963, -.02000823670, -.1599905439$$

Now it appears that the equilibrium point is not stable. What happened? It took us a while to recognize the fallacy.

Answer K is not a Jacobian matrix because the order of taking the derivatives is not the same as the order of the differential equations. If these wayward students had computed

>L:= matrix (4, 4, [-.02, 0, 0, .2, 0, -54,83, 0, -.255, 0, 54,83, -.145, .255, 0, 0, .125, -.22]);

$$L := \begin{bmatrix} -.02 & 0 & 0 & .2 \\ 0 & -54.83 & 0 & -.255 \\ 0 & 54.83 & -.145 & .255 \\ 0 & 0 & .125 & -.22 \end{bmatrix}$$

then we get the same eigenvalues:

>eigenvalues(L);

$$-.02000000000, -54.83058523, -.1461536437, -.2182611306$$

Note that $K = EJ$, where E is

$$E := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and E has determinant -1. Therefore $\det(K) = -\det(J)$, so it cannot be expected that the two matrices have the same eigenvalues. However, $L = EJE^{-1}$, so that L and J are similar with the same characteristic equations and eigenvalues. Given that one of the eigenvalues is very close to 0, there may be some numerical issues to investigate, too.

References

[1] D. Earn, et al, A simple model for complex dynamical transitions in epidemics, *Science* 287 (2000) 667–670.

The article by Edward Aboufadel in the September, 2001 issue of *CMJ* (32 (2001) 279–281) drew the following response from Jacques Laforgue of the Universidad De Oriente in Cumana, Venezuela. It is followed by Aboufadel's reply.

Lively Cities The September 2001 Fallacies, Flaws and Flimflam column dealt with the numerical study of a system of differential equations modeling the dynamics of a disease in cities like London, Liverpool or Baltimore [[1]]. I find that two parameter values used therein are cause for reflection.

One of the equations is

$$\frac{dS(t)}{dt} = b - ri(t)S(t) - dS(t),$$

where t is time, $S(t)$ is the fraction of the population that is susceptible to infection, $i(t)$ is the fraction that is infectious, b and d are birth and death rates, and r is the rate of infection (r is proportional to the probability of an encounter between an infectious person and a susceptible one, per time unit.)

In the text, the values $b = d = 0.02$ were mentioned. This means, for example, that births alone will duplicate the city population in $(\ln 2)/(0.02) \sim 35$ years, which makes sense. However, the time unit used in the system was a day, not a year, and the numbers $b = d = 0.02$ per year were still used. A birth rate of 2% per day would mean a duplication of the city population in thirty-five days. (Should we compare with rabbit performances?)

The value used for the rate of infection was $r = 700$ [per day]. From the differential equation, we can interpret that in a short enough period of time Δt, the fraction of susceptibles getting infected is approximately $\frac{\Delta S}{S} \sim ri\,\Delta t$. Taking, for example, the value of i as the asymptotically stable equilibrium ($i = 0.07834\cdots$) which was given, and converting Δt days into $h/24$ hours, we get $\frac{\Delta S}{S} \sim 2.3h$. So it would take *less than half an hour* for all the susceptibles to get infected. Taking into account that the infectious are less than 8% and the susceptibles are less that 0.04%, one can imagine a very speedy Brownian motion in these (definitely lively) cities.

References

[1] E. Aboufadel, An epidemic of Jacobians, *College Mathematics Journal* 32 (2001) 279–281

Sickly Cities In "Lively Cities", Laforgue presents evidence that something is amiss with the parameters in the SEIR model from "An Epidemic of Jacobians." While his evidence is compelling, he missed what could have been the final nail in the coffin: the equilibrium point! (I regret that the values of the equilibrium point in my original report are out of order; those veterans of *Maple* will understand how this error could occur.) At equilibrium, practically 0% are left who are susceptible, 14% have been exposed to the disease but not infected, 8% are infected, and 78% have recovered. While this might be the appropriate numbers for a broken heart, it appears that at any point in the foreseeable future, 1 out of 12 residents of these so-called lively cities will be quite ill.

In the original *Science* paper, Earn *et al.* considered the situation where r actually varies with time, to take into account seasonal fluctuations due to the length of the school year. The unit of time used was years (*contra* Laforgue) and, in a footnote describing the equations of the SEIR model, it was observed that if r is constant, then there will be a stable equilibrium point. This footnote inspired the problem that I posed to my students, and in the *Science* article, the parameters for latency and infection periods were given in days. Earn, in a private correspondence, says that they stated the latent and infectious periods in days, as is usually done, since a day is the natural unit when discussing those parameters (whereas a year is the natural unit when discussing birth and death rates), but nevertheless adjusted the parameters appropriately before calculating. My students and I, unwittingly celebrating the purpose of this column, did not.

If the calculations are redone by modifying those two parameters, then at equilibrium, there are about 90% recovered from the measles, 10% susceptible, while those exposed or infected are practically nil, a more reasonable result. This equilibrium point is stable.

CHAPTER 7

Combinatorics

7.1 A faulty test question

Joseph G.R. Martinez describes how students in a teaching methods course were frustrated with a word problem that appeared on a competency examination that no one had been able to solve:

> *Mrs. Amos has 28 students in her class. 17 students have brown hair and hazel eyes. 15 students have brown hair. 10 students have hazel eyes. How many students have neither brown hair nor hazel eyes?*

The multiple choice answers were: 5, 6, 7, 10. Martinez writes, "The problem seemed simple enough, but none of the students' attempted solutions made sense, and none yielded answers to match the printed choices. Interestingly, all of the students assumed that the problem was solvable and that the difficulty lay in their own understanding or mathematical skills. After all, this was an important examination devised by expert educators to test the knowledge of novices. It seemed reasonable to assume that the items in the test were error free.

"The students were shocked but intrigued when their professor asked, 'Could there be an error in the question? And if so, what might the error be?' Once given permission to think the unthinkable, almost immediately several students identified the flaw: the number of students with both brown hair and hazel eyes (17) could not exceed the numbers for brown hair (15) and hazel eyes (10) individually. There had to be an error in the figures printed.

"Working by trial and error, the students speculated that the total number with brown hair and hazel eyes could be 7 instead of 17." This would lead to 18 students with either brown hair or hazel eyes and leave 10 students

(one of the options) with neither. "Finding an error where they assumed there could not be an error taught these future teachers an important lesson in critical thinking. It also showed them the value of errors in the learning process: the mathematical reasoning involved in finding the error was much more complex and interesting than solving the original, correct problem would have required."

7.2 Funky Yahtzee

Dale R. Buske writes the following account of how a combinatorial problem might be solved by students:

"A game that I will call *Funky Yahtzee* consists of rolling *six* dice. In a random experiment in which six N-sided dice (each numbered from 1 to N) are rolled, there are N^6 equally likely outcomes. For example, there are ${}_6C_5 \cdot_N P_2$ ways to obtain "five of a kind." It is a fun project to assign student groups an N-value and then the problem of counting how these N^6 outcomes constitute the following 12 disjoint events: *Yahtzee* (six of a kind), five of a kind, four and a pair, four of a kind, two triples, large straight (six consecutive numbers), triple and a pair, three of a kind, three pair, two pair, one pair, and none of these.

"While standard solutions to this problem involve the use of combinations, permutations, and the multiplication rule, consider the following somewhat elegant solution to finding the number of outcomes yielding one pair. With a fictional 10-sided die assigned to them, student group A had already been through the arduous task of computing the number of outcomes yielding two pair, namely ${}_6C_2 \cdot_4 C_2 \cdot_{10} C_2 \cdot_8 P_2 = 226,800$. Rather than this tedious approach in which one has to compute several combinations and multiply, group A realized that obtaining two pair is "twice as difficult" as obtaining one pair. Hence, they argue, there are twice as many outcomes ($2 \cdot 226,800 = 453,600$) yielding one pair as two pair.

"Not to be outdone, student group B (also assigned an N-value of 10) avoided the seemingly insurmountable task of computing the number of outcomes in the two-pair event:

> To determine "Two-Pair", we thought logically. Being that "Two-Pair" is in the middle of "One-Pair" and "Three-Pair", we took the number of outcomes for "One-Pair" and divided it by the number of outcomes from "Three-Pair": $453,600/10,800 = 42$. We then divided that answer in half, which equals 21. We then took the number of outcomes of the "Three-Pair" and multiplied it by 21, giving us the answer of 226,800.

Group B realizes that the answer for "Two-Pair" should be halfway between "One-Pair" and "Three-Pair".

"To my surprise, the groups with other N-values did not construct such poetic solutions. Perhaps the following general computation explains why:

$$\begin{aligned} \textit{Three Pair:} &\quad 15 \cdot_N P_3 \\ \textit{Two Pair:} &\quad 45(N-3) \cdot_N P_3 \\ \textit{One Pair:} &\quad 15(N-4)(N-3) \cdot_N P_3 \end{aligned}$$

7.3 Combination lock

Problem The lock on a safe consists of three wheels, each of which may be set in eight different positions. Because of a defect in the safe mechanism, the door will open if any two of the three wheels are in the correct position. What is the smallest number of combinations which must be tried by someone not knowing the correct combination to guarantee opening the safe?

Solution (by a student) At least 64 are necessary. Let (a, b, c) represent a trial combination, where a, b, c each represent one of the positions of a different wheel. This will pick up any of the following two-wheel combinations: $(a, b, -)$, $(a, -, c)$, $(-, b, c)$. All told, there are 8^2 ways of specifying the positions of two of the three wheels; since there are 3 ways of picking the two wheels, there are $3 \times 8^2 = 192$ possibilities for the positions of two of the three wheels. Since each trial (a, b, c) covers only three of them, we need at least $192/3 = 64$ trials.

However, 64 trials suffice. We make sure that the trials include one for each of the 8^2 possible positions of the first two wheels, so that any combination will match the position of two wheels of exactly one of the trials. □

Comment The difficulty with the argument is that it focuses on duplicating a given two of the three wheels by a given choice of (a, b, c) with a single fixed trial. A given position of two wheels may be picked up by separate trials depending on the state of the third wheel. To clarify this point, let us modify the situation by specifying that each wheel may be set in only two different positions, denoted by 0 and 1. Then it is clear that the trials $(0, 0, 0)$ and $(1, 1, 1)$ will suffice to open the safe.

However, according to the argument adapted to this situation, we would have $3 \times 2^2 = 12$ possibilities for the two wheels and require at least four trials. If, say, $(0, 0, 1)$ were one of these trials, it would cover $(0, 0, -)$, $(0, -, 1)$ and $(-, 0, 1)$. However, it would also pick up $(-, 1, 1)$ if the blank were filled up by 0, but not if it were filled by 1.

Let us return to the given problem, and suppose that each position of a wheel is given by one of the numbers $0, 1, 2, 3, 4, 5, 6, 7$. For the trials, select 16 combinations given by (x, y, z) where $0 \leq x, y, z \leq 3$ and $x + y + z \equiv 0 \pmod 4$ and 16 combinations given by (u, v, w) where $4 \leq u, v, w \leq 7$ and $u + v + w \equiv 0 \pmod 4$. Given any combination (a, b, c), either two wheels have a position coded by numbers from $\{0, 1, 2, 3\}$ or two wheels have a position coded by numbers from $\{4, 5, 6, 7\}$. In the first instance, these two wheels have their positions marked by one of the first sixteen trials, and in the second instance, one of the second sixteen trials.

7.4 Lines of cubes in a block

Problem Tic-tac-toe can be played on a $4 \times 4 \times 4$ cubic block of 64 unit cubes, in which the winner is the first person to select four cubes in a line, either parallel to an edge or along a diagonal. In how many ways can this be done?

Solution 1 The answer is 124. There are 12 4×4 flats, four parallel to each facial plane of the cube; a line of four unit cubes can be selected in 10 ways on each flat, eight parallel to a side and two diagonal. Thus 120 lines can be obtained in this way. In addition, there are four body diagonal lines of the cubic block.

Solution 2 The answer is 76. Coat the $4 \times 4 \times 4$ cubic block with additional unit cubes to form a $6 \times 6 \times 6$ cubic block. Each line of four unit cubes can be continued to a line in the larger block, and each outside cube is the extension of exactly one line of four cubes. Since each line of 4 cubes corresponds to two of these outer cubes, one at each end, the number of lines of four unit cubes is $\frac{1}{2}(6^3 - 4^3) = 76$.

Comment In Solution 1, there is double counting, in that non-diagonal lines in any flat actually belong to two different flats. There are 4 lines of four along a body diagonal, 24 lines along diagonals of one of the flats and $48 = \frac{1}{2}(8 \times 12)$ lines parallel to a side of exactly two of the flats.

7.5 Eight is enough

As readers may know, the Big Ten athletic conference has eleven teams (does this imply that one of the teams is not deemed "big"?). Currently, every team plays eight of the others in football each season, but from time to time, suggestions are made to increase that number. It was reported in an article, *Big Ten coaches say eight is enough*, in the June 10, 2006, issue

of the Lafayette, IN *Journal and Courier*, that "the coaches voted down the idea of adding a ninth conference game to the new 12-game regular season schedule." The article went on: "A ninth league game would mean that Big Ten teams would play every other conference foe but one, instead of the current setup where each team has two league foes missing from their schedule. The coaches fear that playing nine league games would mean that Big Ten teams would beat up on each other and hurt their chances of being ranked high and thus hurting their bowl chances. Plus, that would mean that some teams would play five conference home games while others would play only four."

No one seemed to have noticed that each of eleven teams playing nine games against the others is a mathematical impossibility! Just consider the graph matching the teams that do not play each other.

I am indebted to I.B. Keene for this item.

7.6 Yet another refreshing induction fallacy

The following essay was submitted by Shay Gueron, Department of Mathematics, Technion - Israel Institute of Technology, Haifa, Israel.

Part One

Introduction Paradoxes and fallacies are favorites among mathematics teachers, as well as professional and amateur mathematicians. We know that all triangles are isosceles and that one equals zero; we are aware of other analogous important facts (see [10, 2, 3, 9] for collections, and [6] for an historical view). Mathematical induction is known to be a prolific source for paradoxes and fallacies. This is particularly true since all positive integers are equal [11], since one lighter counterfeit can be identified from any number of seemingly identical coins with only four applications of an equal-arms pan balance [1, 8], and since $1/(1 \times 2) + 1/(2 \times 3) + \cdots + 1/((n-1) \times n) + 1/n = 3/2$ is a valid identity [7].

The above three induction fallacies, as well as many others, revolve around hiding the fact that a certain argument used in the proof fails for some initial (or small) value of the induction parameter. The rest of the argument is correct. However, as the initial validation stage of the induction proof is fallacious, the whole proof is incorrect. We call such fallacies "validation step fallacies".

This note presents yet another induction fallacy, but this one is based on a somewhat different principle. Here is the deal: we propose a problem and offer two solutions. One solution is correct, so you can be sure that the

statement of the problem is true. However, the other solution is flawed, and you are invited to find which one it is — and why.

The problem Consider a convex bounded polyhedron whose faces are triangles. Prove that it is possible to color its edges either red or blue in such a way that the following property holds: one can travel from any vertex to any other vertex while passing only along red edges, and can also do this while passing only along blue edges.

Naming things For convenience, we call any coloring of the edges of a polyhedron, which satisfies the requirements of the problem, a *good coloring*. We call any bounded convex polyhedron (polytope) whose faces are triangles a *triangular polyhedron*.

First proof Consider a convex triangular polyhedron with $n \geq 4$ vertices. We choose an arbitrary vertex C. Suppose that $r \geq 3$ edges emanate from C to vertices $C_1, C_2, \cdots, C_r$. Because of the polyhedral structure, there exists a proper labeling of these vertices, say, $C_1, C_2, \cdots, C_r$ in such a way that $C_1C_2, C_2C_3, \cdots C_{r-1}C_r, C_rC_1$ are edges. We color CC_1 blue, $CC_2, CC_3, \cdots, CC_r$ red, C_1C_2 red and $C_2C_3, C_3C_4, \cdots, C_rC_1$ blue.

We now proceed by rounds: we choose any triangular face F which has either one or two colored edges. We consider the following two cases:

(a) If F has only one colored edge, we color the remaining two edges with different colors: one red and one blue. Note that before this coloring, F had precisely one vertex which was yet unreachable by colored paths. We call it a "new vertex."

(b) If F has two colored edges, we color the third edge with any color. Note that all of the vertices of F were already reachable by colored paths even before the coloring of this third edge.

From our coloring it follows that in case (a) we can reach the new vertex from C by both red and blue paths. In case (b) no new vertex is reached. At the end of this round, the vertices that belong to completely colored faces can be reached from C by both red and blue paths.

We continue with these rounds inductively: at each step we seek a new face with either one or two colored edges and repeat (a) or (b), accordingly. This procedure preserves the good coloring property of the subset of vertices that belong to completely colored faces. We conclude after all edges have been colored. This must occur within a finite number of steps. Otherwise, we would have two *disjoint* subsets of faces: one which is completely colored and the other which is completely uncolored. This contradicts the polyhedral structure. At this point, we are done.

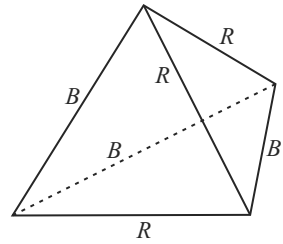


Figure 1. A good coloring of a tetrahedron ("R" stands for red and "B" for blue)

Second proof We prove that a good coloring exists by induction on n, the number of vertices of the polyhedron. First, we verify that the statement is true for $n = 4$, that is, for a tetrahedron. A possible coloring is given in Figure 1.

Suppose that the statement holds for any triangular polyhedron with n vertices, and consider the triangular polyhedron P_{n+1} whose $n+1$ vertices are denoted by $C_1, C_2, \cdots, C_n, C_{n+1}$. We identify the vertices C_n and C_{n+1} (*i.e.,* "merge" them into one), together with the corresponding edges and faces, in order to obtain a new polyhedron, denoted P_n, whose n vertices are $C_1, C_2, \cdots, C_n$. See Figure 2 for an illustration of this procedure.

By the induction assumption, there exists a good coloring for the polyhedron P_n, so we use it. We now *split* the two identified vertices C_{n+1} and

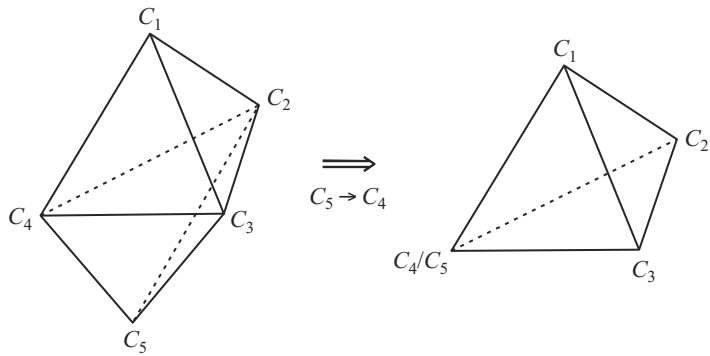


Figure 2. The procedure of identifying and splitting vertices, demonstrated for a triangular polyhedron with $n = 5$ vertices. The vertex C_5 is identified with the vertex C_4. The corresponding edges are changed as follows: C_5C_2 is identified with C_4C_2, C_5C_3 is identified with C_4C_3, and C_5C_4 disappears. The corresponding faces are changed as follows: $C_5C_2C_3$ is identified with $C_4C_2C_3$, and $C_5C_4C_3$ and $C_5C_2C_4$ disappear. This leaves us with a reduced polyhedron having $n = 4$ vertices. When C_4 is split and C_5 is reinstated together with the corresponding edges and faces, we reproduce our original polyhedron.

C_n, and get P_{n+1} back in business (see Figure 2). To complete the coloring of P_{n+1} we need to color the yet uncolored edges. These are the edges C_kC_{n+1} $(1 \le k \le n-1)$, which before the splitting were the edges C_kC_n. Here is how we color them: for $1 \le k \le n-1$ we color the edge C_kC_{n+1} in the color of C_kC_n. If, with no loss of generality, $C_{n-1}C_n$ is red, we color the edge $C_{n-1}C_{n+1}$ red and the edge C_nC_{n+1} blue.

To prove that we end up with a good coloring, it remains to show that we can reach the vertex C_{n+1} from any vertex C_k by both red and blue paths. This is true since we can reach C_n from C_k in either red or blue paths, and then can continue from C_n to C_{n+1} by either a red path $(C_nC_{n-1} \to C_{n-1}C_{n+1})$ or a blue path (C_nC_{n+1}). At that point, we are done.

Statement The statement of the problem is true, and one of these two proofs is correct. The other proof is not. Which is the false proof, and where is the flaw? In the next part, the answer will be given.

Part Two

Unveiling the fallacy The second proof is the flawed one. The problem lies in the induction step, but is *not* due to the "classical" type of pitfall that occurs only for small/initial n. In fact, the validation step (for $n = 4$) is correct. Furthermore, the coloring scheme performed after splitting the extra vertex, as described in the transition stage, is correct. The failure is rather due to the polyhedral geometry of the problem, as there is a "slight" difficulty with applying the induction assumption to the "polyhedron" P_n: when identifying C_n and C_{n+1}, the resulting object, P_n, does not have to be a convex triangular polyhedron. In fact, it is not necessarily a polyhedron at all! Figure 2, which was brought for illustrating the procedure, is deceiving. Figure 3 illustrates why.

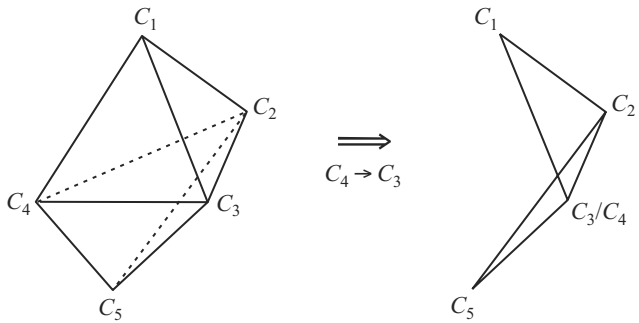


Figure 3. Identifying C_n and C_{n+1} does not necessarily yield a polyhedron.

It follows that the inductive reduction (identifying two edges) does not reduce the discussion to an object that belongs to the family upon which the induction assumption was made. Consequently, the induction assumption cannot be applied, and the proof is erroneous.

Concluding remarks It is interesting to note that, although we referred to the "polyhedral geometry of the problem," the problem is purely a combinatorial one. It actually deals with coloring the graph of a triangular polyhedron. Some properties of such graphs were implicitly used in the first proof. The combinatorial approach suppresses the three dimensional geometric (difficult) flavor of the problem, that was the source of the trap in the second proof. The general characterization of graphs of convex polyhedra (not necessarily triangular) is the following (Steinitz' theorem, see (3): a graph with at least three vertices is the graph of some three dimensional convex polytope (bounded convex polyhedron) if and only if it is planar and 3-connected (*i.e.*, it remains connected even after the removal of any two of its vertices). However it is not difficult to show that this general characterization is not absolutely necessary here: our problem can be formulated (why?) in terms of coloring the edges of a triangulation in the plane (*i.e.*, a non-self-intersection graph in the plane whose faces are triangles). Induction can be successfully applied to such triangulations. This leads to the following extension problems, that are left to the reader for further study:

1. Construct a valid proof by induction.
2. Generalize the coloring problem to other types of polyhedra.
3. Find how many different good colorings exist for a given triangular polyhedron.

Finally, an historical note: the problem discussed here appeared in the Israel-Hungary Bi-National Mathematics Competition in 1995 [4]. The flawed proof described here was indeed attempted by some of the contestants. Since then, I have been using it as an "educational hoax" in the training of the Israeli team for the International Mathematical Olympiads, and in other classes for advanced students **[5]**. As a trap that is introduced after a session on mathematical induction and its applications for problem solving — it never fails.

Acknowledgment. I thank Ron Holzman and Itai Shafrir for many helpful discussions and suggestions. This research was supported by the Technion V.P.R. Fund and by the Fund for the Promotion of Research at the Technion.

Editorial comment Readers may be interested in comparing this situation with [13].

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CHAPTER 8

Probability and Statistics

8.1 Chance of meeting

Problem A and B are standing at the respective points $(0, 0)$ and $(2, 1)$ in a square grid. At the same instant, with the same speed, each walks towards the position of the other. A moves only to the right and up, while B moves only to the left and down; both must travel on grid lines (at least one of the coordinates is an integer). What is the probability that they will meet?

Solution 1 Each person has three possible paths, according as the route goes along the segments, $x = 0$, $x = 1$ and $x = 2$. They will meet if and only if they pick the same segment, and thus travel along the same route in opposite directions. Thus, they will meet with probability $1/3$. □

Solution 2 In the time taken to travel unit distance, A has arrived at either $(0, 1)$ or $(1, 0)$, each with probability $1/2$, and B has arrived at either $(1, 1)$ or $(2, 0)$, each with probability $1/2$. They will meet under the following circumstances:

(a) A is at $(0, 1)$ and B is at $(1, 1)$, and B decides to continue towards $(0, 1)$ rather than go to $(1, 0)$. This occurs with probability $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$.

(b) A is at $(1, 0)$ and B is at $(2, 0)$ and A decides to continue towards $(2, 0)$ rather than go to $(1, 1)$. This occurs with probability $\frac{1}{8}$.

(c) A is at $(1, 0)$ and B is at $(1, 1)$, and each decides to move along $x = 1$ rather than continue in their original directions. This occurs with probability $(\frac{1}{2})^4 = \frac{1}{16}$.

Thus, the probability of their meeting is $\frac{1}{8} + \frac{1}{8} + \frac{1}{16} = \frac{5}{16}$. □

Comment Actually, both solutions are correct. Each of the solutions presumes a particular way of formulating the randomness. In the first solution, the three possible paths are selected with equal probability. In the second solution, where there is a choice of direction within the course of the journey, each choice is equally likely. In the latter case, the path

$$(0,0) \to (0,1) \to (1,1) \to (2,1)$$

has probability $\frac{1}{2}$ while each of the paths

$$(0,0) \to (1,0) \to (1,1) \to (2,1) \text{ and } (0,0) \to (1,0) \to (2,0) \to (2,1)$$

has probability $\frac{1}{4}$.

8.2 Not quite safe

Paul Stockmeyer related his favorite anecdote, which was related to him by Allen Schwenk when he was teaching at the US Naval Academy many years ago. It seems that the Navy had a new surface-to-air missile that could shoot down an attacking aircraft with probability $1/3$. Some top Navy officer then claimed that shooting off three such missiles at an attacking aircraft [presumably with the usual assumptions of independence] would surely destroy the attacker. Schwenk asked his mathematics students to critique this officer's reasoning. One midshipman whipped out his calculator and declared, "Let's see. The probability that the first missile does the job is 0.3333, same for the second and same again for the third. Adding these together, we get 0.9999, so the officer is wrong; there is still a small chance that the attacking aircraft survives unscathed." Just think, notes Stockmeyer, that student might himself be a top US navy officer, defending North America from attack.

8.3 A wrong route to a right answer

Exercise Show that, if Y is a random variable with a geometric distribution, the expected value of

$$\begin{aligned} \mu &= E(Y) \\ &= \sum_{k=1}^{\infty} kpq^{k-1} \\ &= \frac{1}{p}. \end{aligned}$$

Solution Note that $p + q = 1$.

$$\int E(Y)dq = \int \sum_{k=1}^{\infty} kpq^{k-1}dq = p\sum_{k=0}^{\infty}\int kq^{k-1}dq$$
$$= p\sum_{k=0}^{\infty}(q^k + c) = p\sum_{k=0}^{\infty} q^k + c_1 = p\frac{1}{1-q} + c_1.$$

Therefore

$$E(Y) = \frac{d}{dq}\left(\int E(Y)dq\right) = \frac{d}{dq}\left(p\frac{1}{1-q} + c_1\right)$$
$$= p\frac{d}{dq}\left(\frac{1}{1-q}\right) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}. \qquad \square$$

This item was submitted by Yongzhi Yang.

8.4 The game of Hol-Tzak

Sung Soo Kim writes about the Korean game *Hol-Tzak*. The name involves the Korean words for odd number and even number; the game is played between two players. Player A holds a (usually random) number of marbles in her closed hand, and B is required to guess whether the number is odd or even. If B is correct, he gains a point; otherwise the point goes to A. He wonders whether the fairness of the game is somehow related to the identity

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{2} + \binom{n}{5} + \cdots.$$

He recounts that he understood this perplexing situation only after he recalled a game *Hana-Dool-Set* (literally *One-Two-Three*) played among three players. A player holds some marbles, and the other two make different guesses as to the remainder when the number of marbles is divided by 3. This game can be played very fairly, but in this case we do not have the validity of

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots = \binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \cdots$$
$$= \binom{n}{2} + \binom{n}{5} + \binom{n}{8} + \cdots.$$

8.5 Lewis Carroll's obtuse problem

Ruma Falk wrote an article entitled *Lewis Carroll's obtuse problem*, published in *Teaching statistics* 23:3 (Autumn, 2001), 72–75. She discussed the Pillow Problem No. 58 from the collection published in 1895 by Lewis Carroll:

> Three points are taken at random on an infinite Plane. Find the chance of their being the vertices of an obtuse-angled Triangle.

The following solution is provided. Take the longest side of the triangle, denoted by AB, and construct a semi-circle AFB with it as diameter. On the same side, trace arcs AEC and BDC both with radius AB and with centers at A and B that meet at C. Then the third vertex of the triangle cannot fall outside of the figure $AECDB$. If it falls inside the semi-circle, then the triangle is obtuse, so that the desired probability is the ratio of the area of the semi-circle to that of the figure $AECDB$, namely

$$\frac{3\pi}{8\pi - 6\sqrt{3}} \quad \text{or about} \quad 0.6394.$$

Falk claims that this is not the solution to the problem posed by Carroll, but of another one in which two arbitrary points are given and one seeks the probability of selecting a third point giving an obtuse triangle among those triangles with the two points bounding the longest side.

Another solution is to take two points A and B as two vertices of the triangle, not necessarily bounding the longest side, and then locate C in a half-plane bounded by the line AB. Unless A lies in that part of a semi-infinite strip bounded by AB outside a semi-circle of diameter AB, the triangle is obtuse, so that the probability of getting an obtuse triangle is equal to 1. The referee of the paper then stirred the pot by arguing that in this situation, the probability that CAB, in particular, is obtuse is $\frac{1}{2}$. By symmetry, the probability that each of the other two angles is obtuse is $\frac{1}{2}$. Since these events are mutually exclusive (a triangle cannot have two obtuse angles), the desired probability is in fact $\frac{3}{2}$!

The difficulty with such problems is the infinitude of the sample space and what it means to have a uniform distribution for such a space. It seems that to cut through the confusion, one has to actually envisage performing the experiment. You are then forced to make a decision as to what exactly is being randomized and this directs the solution. What does the reader think?

I am indebted to Richard K. Guy who responded with a reference to his article, "There are three times as many obtuse-angled triangles as there are

acute-angled ones," in *Mathematics Magazine* 66 (1993), 175–179. He describes various arguments giving different values of the probability before settling on five arguments to justify the assertion of the title. For example, any triangle along with its orthocenter determines a set of four orthocentric points, any of which is the orthocenter of the triangle determined by the other three. Thus, triangles occur in clusters of four determined by taking sets of three of four orthocentric points. Three in each cluster have an obtuse angle.

David E. Dobbs and Jill L. Zarestky wrote a paper several years ago entitled, "Acute random triangles" that appeared in *Nieuw Archief voor Wiskunde* (4) 15:3 (November, 1997), 141–162. Dobbs writes that they showed that "by introducing parameters into several known approaches to the problem of random acute triangles and using uniform distributions, one obtains answers for probabilities that are not unique and actually fill up intervals of real numbers. In particular, Remark 4.2 of that paper shows that in the 'doubly pinned' case mentioned by Falk, the probability of a random triangle being acute can be any positive real number in a closed interval $[0, \alpha]$, where α is an irrational number that is approximately equal to 0.360617, a value that is quite close to 1 minus Falk's value of 0.6394.

"The discussion [in *CMJ*] also mentioned models involving semi-infinite strips. There is considerable discussion of such models and their history in my paper with Zarestky: see our Remark 3.3(b) for a related interval of probability values. You may have opened opened up a quagmire, given the various claims of 'bias' that have found their way into the literature on this problem. See especially, the bottom half of page 142 of my paper with Zarestky, as well as the papers of Guy [cited above] and Portnoy [*Statist. Science* 9 (1994), 279–284] that are cited there."

8.6 Blind sampling

Jerry M. Stelmaszak points out the surprising solution to the following problem: *A large bowl contains b black and r red balls. A ball is selected at random and then discarded. A second ball is then selected at random. What is the probability that the second ball is black?* Using conditional probabilities, we find that the required probability is equal to

$$\frac{b}{b+r} \times \frac{b-1}{b+r-1} + \frac{r}{b+r} \times \frac{b}{b+r-1}$$

which reduces to $b/(b+r)$, the probability that the first ball drawn is black. This, he says, is one of those situations in which mathematics goes against common sense.

James C. Alexander comments: "The result may seem against common sense from one point of view, but, suppose all the balls are chosen with a blindfold on and laid out in order in a row. Point to any ball — first, second, ..., last. The probability it is black is just $b/(b+r)$ independently of the position. This is a generalization of 'drawing straws'. Suppose n people have to choose one for some unpleasant task. They draw straws — n straws are presented in such a manner that their lengths cannot be seen. $n-1$ are of equal length; one is shorter. Individuals draw straws one at a time — short straw loses. Is it best to draw first? last? somewhere else? Answer: it doesn't matter. The probability is $1/n$. The ball question is the generalization that there are b unpleasant tasks for $b+r$ people." Pete Johnson, of Auburn University in Alabama, also not surprised, points out that before the drawing, each of the balls has the same probability $(1/(b+r))$ of being drawn *second*, so that the probability of the second ball being one of the b black ones is $b/(b+r)$. In a similar way, the same probability obtains for the last of k balls drawn being black. G. Thomas Sallee noted that "the result is easier to accept if you consider drawing two balls out of the urn simultaneously and arbitrarily choose one of them to be *first* and the other one *second*. In this case, clearly each ball has the same probability of being black."

Raymond N. Greenwell also disagrees that the mathematics goes against common sense, but he provides an extra dimension to one's intuitive take. "When students whom I present with this example seem uneasy with the answer, I ask them whether they think the probability that a ball is black is higher than it was at first, given that they don't know the color of the first ball selected. They invariably respond no. So I ask whether the probability is lower, and they again respond no. I point out that this doesn't leave too many choices! This is not a situation in which mathematics goes against common sense, but one in which 'common sense' goes against common sense."

Stan Lipovetsky remarks that, discarding the first drawn balls without noting their colors gives us no information and we cannot improve the original estimate of $b/(b+r)$ for the probability. He adds, "The [detailed] evaluations show pretty well that the conditional probability, being averaged over all possible situations, coincides with unconditional probability. It could seem that we are considering the scheme without returning the balls, although such an average of conditional probabilities corresponds to the scheme with returning the drawn balls."

Barthel Wayne Huff makes the same point about not getting any information from the earlier draws. The result can be used in class to help students

develop their understanding. He notes that, when all balls are drawn from the urn, drawing the last ball is equivalent to drawing a ball from the discard urn and returning it to the original urn, so that the probability that the last ball is black equals the probability that the first drawn ball is black. It is now plausible that the probability is the same regardless where the ball under consideration is in the sequence of drawn balls. Huff imagines the equivalent experiment of pouring the $b + r$ balls into a trough with $b + r$ cavities labeled $1, 2, \cdots, b + r$ and identifying the ball in the kth cavity with the kth drawn ball. He concludes his comments with:

> Our observation that $P(B_k) = b/(b + r)$ [for the probability that the kth drawn ball is black] and our ability to calculate such probabilities in other ways now leads to results that we might not have otherwise suspected. For example, if $b = 10 = r$ and A_j is the event that j black balls were obtained and discarded in the first three draws, then we can use the law of total probabilities to obtain
>
> $$\begin{aligned}\frac{1}{2} &= P(B_4)\\ &= P(B_4/A_0)P(A_0) + P(B_4/A_1)P(A_1)\\ &\quad + P(B_4/A_2)P(A_2) + P(B_4/A_3)P(A_3)\\ &= \frac{10}{17}\cdot\frac{\binom{10}{3}\binom{10}{0}}{\binom{20}{3}} + \frac{9}{17}\cdot\frac{\binom{10}{2}\binom{10}{1}}{\binom{20}{3}} + \frac{8}{17}\cdot\frac{\binom{10}{1}\binom{10}{2}}{\binom{20}{3}} + \frac{7}{17}\cdot\frac{\binom{10}{0}\binom{10}{3}}{\binom{20}{3}}.\end{aligned}$$
>
> The reader is invited to find and verify general forms of such equalities.

8.7 $\pi = 3$

Problem Suppose points are chosen at random on the circumference of the upper half of the unit circle $x^2 + y^2 = 1$. What will be their expected distance from the point $(1, 0)$?

Solution 1 The coordinates of the point are $(\cos\theta, \sin\theta)$ with $0 \le \theta \le \pi$. Since the distance between this point and $(1, 0)$ is

$$\sqrt{2(1 - \cos\theta)} = 2\sin(\theta/2),$$

the desired average length is

$$\frac{1}{\pi}\int_0^\pi \sqrt{2(1 - \cos\theta)}d\theta = \frac{4}{\pi}. \qquad \square$$

Solution 2 The coordinates of the point are $(x, \sqrt{1-x^2})$, with $-1 \leq x \leq 1$. Since the distance between this point and $(1, 0)$ is $\sqrt{2(1-x)}$ and x ranges over an interval of length 2, the desired average length is

$$\frac{1}{2}\int_{-1}^{1} \sqrt{2(1-x)}dx = \frac{4}{3}. \qquad \square$$

Thus, $4/\pi = 4/3$ and so $\pi = 3$. Contributed by Frank Burk.

Comment The answer depends on the probability distribution inherent in the integral that is set up. In the first situation, a uniform distribution is assumed for the angle θ, while, in the second, for x. The two are not equivalent.

8.8 A snafu

Kenneth Schilling asks us to consider a pair (X, Y) of independent standard normal random variables with the joint density function

$$f(x, y) = \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}$$

for $x, y \in \mathbf{R}$. Change to polar coordinates (R, Θ). Then $R = \sqrt{X^2 + Y^2}$ and Θ is the argument of (X, Y) in the interval $[0, 2\pi)$; that is, Θ is the angle that the ray from the origin through (X, Y) makes with the positive $x-$axis.

The problem is to find the distribution of R. It is easy to see that R and Θ are independent, so that for $r_0 \geq 0$,

$$P(R \leq r_0) = P(R \leq r_0 | \Theta = 0).$$

Now $\Theta = 0$ exactly when $X \geq 0$ and $Y = 0$, so that

$$P(R \leq r_0 | \Theta = 0) = P(R \leq r_0 | X \geq 0 \text{ and } Y = 0).$$

When $X \geq 0$ and $Y = 0$, then $X = R$ and

$$P(R \leq r_0 | X \geq 0 \text{ and } Y = 0) = P(X \leq r_0 | X \geq 0 \text{ and } Y = 0).$$

Since X and Y are independent,

$$P(X \leq r_0 | X \geq 0 \text{ and } Y = 0) = P(X \leq r_0 | X \geq 0).$$

This probability is easy to compute. Putting these equalities together, we conclude that

$$P(R \leq r_0) = 2\int_0^{r_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx.$$

Of course, this is false. By changing to polar coordinates in the usual way, we have that

$$P(R \leq r_0) = \int_0^{r_0}\int_0^{2\pi} \frac{1}{2\pi} e^{-\frac{1}{2}r^2} r\, dr d\theta = 1 - e^{-\frac{1}{2}r_0^2}.$$

Where is the flaw in the argument?

8.9 A certain match

In the book *Developing Creative and Critical Thinking*, by Robert Boostram, Linda Wagner found a passage that discussed the duplication of birthdays in a group of thirty people. However, it departed from the usual treatment and came to an astounding conclusion.

In essence, the book asks, *Would you be surprised to find that two of the thirty people in your class have the same birthday?* Before answering the question, the reader is urged to figure out the probability. And this is how it is done!

Suppose that there were only two people — you and one other person. The chance that you and the other person have the same birthday is approximately 1 in 366. Now add a third person. The chance that the birthday of the third person will match yours or the other person's is 2 in 366. Continuing in this way, you find eventually that the chance that the thirtieth person will match up with the twenty-nine others is 29 in 366. Therefore, the probability of two people in a group of thirty having the same birthday is (wait for it!)

$$\frac{1}{366} + \frac{2}{366} + \cdots + \frac{29}{366} = \frac{435}{366}.$$

Even death and taxes are not this certain.

Another way of formulating this solution is to note that there are $\binom{30}{2}$ possible pairs and each pair has a probability of $\frac{1}{366}$ of matching, so the total probability is $\frac{1}{366}\binom{30}{2}$.

Dale K. Hathaway finds that there "is a glimmer of potential insight in the approach." He continues:

> The quantity $\binom{30}{2} = 435$ is the number of opportunities for a match, the number of pairs of individuals for which a match is pos-

sible. Frederick Mosteller in an article "Understanding the Birthday Problem" in the May, 1962, issue of the *Mathematics Teacher* showed that the number of opportunities can be used to find out if the probability of at least one match exceeds 50% or not. Mosteller related the birthday problem to what he called the birthmate problem, the problem of a match of a specific birthday. In his article, he showed that the number of people needed in the birthmate problem, n, for a specific probability of a match is approximately equal to the number $\binom{m}{2}$ of opportunities needed for the same probability of a match in the birthday problem with m people. For the typical birthday problem context, requesting the smallest number of people to give a better than 50% chance of a match, the number of opportunities needed is $\binom{23}{2} = 253$.

This opportunities perspective provides an intuitive approach that can be extended to other birthday generalizations. For example, one variation that has been developed for the birthday problem is the boys and girls birthday problem, where we have a group of n girls and n boys and we want to know how large does n need to be to give a better than 50% chance of a match of a birthday of a girl and a boy. While the original solution to this problem used Stirling numbers of the second kind, the opportunities approach gives a nice intuitive solution. For a group of n girls and n boys, each boy has n chances to match his birthday. Since there are n boys, there are a total of n^2 opportunities for a match. Needing at least 253 opportunities to give a probability that exceeds 50% we easily find $n = 16$, which agrees with the answer found through the Stirling number approach. A few other generalizations approached from the opportunities approach are given in the article "A Bonanza of Birthday Bewilderments," by D. Hathaway in the February, 2001, issue of *Math Horizons*.

8.10 Proofiness

Readers may recall that the 2008 Minnesota Senate race was a cliffhanger that took some time to resolve. This letter from Brian Zack of Princeton, NJ, that appeared in the Sunday, October 3, 2010, issue of the *New York Times Book Review*, discussed how such close races might be decided.

Steven Strogatz perpetuates a presumably unwitting example of "proofiness" in his review of Charles Seife's book of the same name ("Fibbing With Numbers," Sept. 9). Strogatz cites Seife's assertion

that the winner of the "astonishing close 2008 Minnesota Senate race between Norm Coleman and Al Franken . . . should have been decided by a coin flip; anything else is disestimation, considering that the observed errors in counting the votes were always much larger than the number of votes separating the two candidates."

Well, no. Although the margin of victory falls within the margin of error — and assuming no systematic bias in the error — the fact that one candidate is ahead in the actual count indicates that the probability of that candidate being the true winner is greater than 50 percent, if only by a little. And that little means that awarding the victory to that candidate has a greater likelihood of picking the true winner than flipping a coin.

If proofiness is "the art of using bogus mathematical arguments to prove something that you know in your heart is true — even when it's not," Strogatz and Seife seem to have succumbed to it in their desire to multiply examples of deceptive mathematics. Congratulations, Senator Franken.

8.11 Teenagers, Sex and Accidents

Here is an essay by Joseph G. Eisenhauer of Canisius College, Buffalo, NY that makes the case that conventional opinion is not always sound.

Ask a classroom full of teenagers whether young men or young women are worse drivers, and one is likely to get a nearly unanimous response: young men. Indeed, it is a widely held belief. And teens are acutely aware of the substantial difference in automobile liability insurance premiums paid by members of the two sexes, which seems to lend validity to the stereotype of the reckless young male. The widespread perception that boys are more careless or accident-prone also seems to be corroborated by news reports such as the following:

Sixteen-year-old boys are still the most risky drivers on the road, but the girls are gaining. For every 1,000 licensed 16-year-old girls, 175 got in car accidents in 2000, according to federal accident data. That's up 9 percent from 1990, when 160 girls crashed per 1,000 drivers. Accidents for 16-year-old boys decreased slightly during the same period, from 216 to 210 per 1,000 drivers. . . . Insurance industry statistics show girls 16 to 19 are driving 70 percent more than in 1975, averaging 6,870 miles a year. Teen boys are driving 16 percent more, averaging 8,200 miles a year. That means parents are having to pay more to insure their teen daughters. State Farm Insurance, the nation's largest auto insurer, charged 16- to 20-year-old

males 61 percent more than females in 1985. Now the difference has fallen to 41 percent [1].

When presented with such evidence, astute students often observe that, although the report mixes age ranges and time periods, there seems to be an injustice, or at least an incongruity: the accident involvement rate for 16-year-old males is 20 percent higher than that of 16-year-old females, yet young males pay 41 percent more for insurance. But there is more: though the report seems to support the notion that girls are generally better drivers, ironically, the fallacy can be partly debunked from information in the report itself. Teenage males drive nearly 20 percent more miles per year; it is this greater exposure, rather than poor driving habits, that explains the difference in accident involvement rates. Put another way, the only fair comparison is between accident rates per mile; on this basis, the gender differential largely disappears. Indeed, a recent study by Allan F. Williams [2] of the Insurance Institute for Highway Safety, using 1995 data, show that teenage males actually have slightly *lower* accident rates per mile than teenage females: 17 crashes per million miles traveled for the former, compared with 18 crashes per million miles for the latter.

It is clear, then, that teenage boys are not more reckless than girls — or rather, girls are not more "wreckless" than boys. Insurance companies merely use gender as a proxy for mileage — a practice that could potentially be avoided by using automobile odometer readings rather than sex to price liability insurance.

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CHAPTER 9

Complex Analysis

9.1 The square root of -1 is real.

Teik-Cheng Lim provides the following development to show that $\sqrt{-1}$ is real.

Start with $\sqrt{-1} = 1 + (\sqrt{-1} - 1)$ and manipulate to obtain

$$\begin{aligned}\sqrt{-1} &= 1 + \frac{(\sqrt{-1}-1)(\sqrt{-1}+1)}{(\sqrt{-1}+1)} = 1 - \frac{2}{(1+\sqrt{-1})} \\ &= 1 - \frac{2}{2 + (\sqrt{-1}-1)} \,.\end{aligned}$$

Iterating this process leads to the beautiful continued fraction

$$\sqrt{-1} = 1 - \cfrac{2}{2 - \cfrac{2}{2 - \cfrac{2}{2 - \cfrac{2}{2 - \cfrac{2}{2 - \cdots}}}}} \,.$$

9.2 An imaginary absolute value?

A very bright student calculated the square of the modulus of a complex number $z = a + b\sqrt{-1}$ thus:

$$\begin{aligned}|z|^2 = z\overline{z} &= (a + b\sqrt{-1})(a - b\sqrt{-1}) \\ &= a^2 - b^2\sqrt{-1}\sqrt{-1} = a^2 - b^2\sqrt{(-1)(-1)} \\ &= a^2 - b^2\sqrt{1} = a^2 - b^2 \,,\end{aligned}$$

so that when $b > a > 0$, then $|z|^2$ is negative. Peter M. Jarvis and Paul S. Shuette, the contributors of this item, emphasize the care needed in defining

roots of negative numbers and warn about the pitfalls of "defining" i as $\sqrt{-1}$.

9.3 A counterexample to Liouville's Theorem

Mark Lynch provides a conundrum that causes his beginning students of complex variables a great deal of consternation. Let $f(z) = \cos z + i \sin z$ be defined on the set of complex numbers. Since $\cos z$ and $\sin z$ are entire, then so is $f(z)$. Also, the modulus of $f(z)$ is equal to

$$\begin{aligned}\sqrt{f(z) \cdot \overline{f(z)}} &= \sqrt{(\cos z + i \sin z)(\cos z - i \sin z)} \\ &= \sqrt{\cos^2 z + \sin^2 z} = 1,\end{aligned}$$

so that $f(z)$ is bounded. Therefore, by Liouville's theorem, $f(z)$ is constant. □

Comment The calculation of the complex conjugate of $f(z)$ is not correct when $\cos z$ and $\sin z$ are not real. This can be seen from the relations

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$$

and

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$

when x and y are real. Indeed, it can be checked that $f(z)$ is not bounded on $\mathbf{C}$.

9.4 The addition formula for sine

Juan Tolosa asked his students to prove that the addition formula for $\sin z$ remained valid in complex variables. Here is one student's justification:

From the distributive property,

$$\begin{aligned}\sin(z + w) &= \sin z + \sin w \\ &= \frac{e^{iz} - e^{-iz}}{2i} + \frac{e^{iw} - e^{-iw}}{2i} \quad \text{(from the definition)} \\ &= \frac{2i \sin z}{2i} + \frac{2i \sin w}{2i} \\ &= \frac{(\sin 2z)i}{2i} + \frac{(\sin 2w)i}{2i} \\ &= \sin z \cos z + \sin w \cos w \ .\end{aligned}$$

Now from the commutative property,

$$\sin z \cos z + \sin w \cos w = \sin z \cos w + \cos z \sin w \ . \qquad \square$$

9.5 Fourier analysis is trivial

An example from Peter M. Jarvis illustrates the need for caution when dealing with exponents in the complex field. Let $f(\theta)$ be a $2\pi-$periodic, integrable function. Then its nth Fourier coefficient is given by

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta \\
&= \int_0^1 f(2\pi t)e^{-2\pi int}dt \\
&= \int_0^1 f(2\pi t)[e^{-2\pi in}]^t dt \\
&= \int_0^1 f(2\pi t)[1]^t dt = \int_0^1 f(2\pi t)dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta = \hat{f}(0) \ ,
\end{aligned}$$

so that all Fourier coefficients are the same.

Comment Note that $e^{2\pi int} = \cos 2\pi nt + i \sin 2\pi nt$ is not equal to 1 when t is not an integer. The exponential law $(a^b)^c$ needs careful handling for complex numbers because of multivalued functions.

9.6 Maximum distance between circumference points

Evangelos N. Panagiotou teaches mathematics at the "Varvakion" experimental Lyceum in Athens, Greece. He sends a solution written by one of this students.

Exercise The complex numbers z and w satisfy the conditions

$$|z + 1 - i| \leq 2 \qquad \text{and} \qquad |w - 3 + 2i| \leq 3 \ .$$

Find the maximum value of $|z - w|$.

Solution By the triangle inequality

$$|z| - |1-i| \le ||z| - |1-i|| \le |z+1-i| \le 2\,,$$

whereupon

$$|z| \le 2 + |1-i| \Leftrightarrow |z| \le 2 + \sqrt{2}\,.$$

Similarly,

$$|w| - |3-2i| \le ||w| - |3-2i|| \le |w-3+2i| \le 3$$

implies that

$$|w| \le 3 + |3-2i| \Leftrightarrow |w| \le 3 + \sqrt{13}\,.$$

Hence

$$|z-w| \le |z| + |w| \le 5 + \sqrt{13} + \sqrt{2}\,.$$

Thus, the maximum value of $|z-w|$ is $5 + \sqrt{13} + \sqrt{2}$. □

The purported maximum is about 10.01. The points $1-i$ and $-3+2i$ are at distance 5 in the complex plane and the two discs described in the exercises are tangent. It follows that the correct answer is 10.

9.7 Strange dependence

Ollie Nanyes reports that a complex variables class had to verify that $\int_\gamma \log z\,dz = 2\pi i$, where γ is the standard unit circle taken in the counterclockwise direction and where the principal value of the logarithm with $\log 1 = 0$ is taken. It was noted that the value of the integral is independent of the base point on the circle γ. This was problem 403 from the book *A Collection of Problems on Complex Analysis* by L.I. Volkovyskii, G.L. Lunts and I.G. Aramanovich (translated by J. Berry), Dover, 1991.

On the final examination, students had to discuss $\int_\gamma \frac{\log z}{z} dz$, where the symbols are as above. One student came up with the following argument. Parametrize γ by $z(t) = e^{it}$ and $dz = ie^{it}dt$, and obtain

$$\begin{aligned}\int_a^{a+2\pi} \frac{\log e^{it}}{e^{it}} ie^{it}dt &= \int_a^{a+2\pi} i^2 t\,dt \\ &= -\frac{1}{2}\big((a+2\pi)^2 - a^2\big) = -2\pi(\pi + a)\,,\end{aligned}$$

which, of course depends on a, or on the starting point on the circle for the integral. What gives?

9.8 Integrating around a closed contour

Dale Buske presents his students with the following complex variables problem designed to assess their understanding of the Cauchy-Goursat Theorem concerning the integration of an analytic function along a closed contour. Since the function of the problem is analytic on and inside the square, the integral vanishes.

Problem Compute the value of

$$\int_C \frac{z}{z^2+2}dz$$

where C is the negatively oriented square defined by the lines $x = \pm 1$ and $y = \pm 1$.

Solution (by a student)

$$\int_C \frac{z}{z^2+2}dz = \int_{C_1} \frac{z}{z^2+2}dz$$

where C_1 is the circle $|z| = \sqrt{2}$. Justification for this is clear since the contour can be deformed so that the square is inscribed in the circle. Parametrize the contour C_1 by

$$z(\theta) = \sqrt{2}e^{i\theta}, \qquad (0 \le \theta \le 2\pi).$$

Then $z'(\theta) = \sqrt{2}i\,e^{i\theta}$. Also, since $|z(\theta)|^2 = 2$, we have that $z(\theta)^2 + 2 = 4$, so that

$$\begin{aligned}\int_{C_1} \frac{z}{z^2+2}dz &= -\int_0^{2\pi} \frac{\sqrt{2}e^{i\theta}}{4}\sqrt{2}i\,e^{i\theta}d\theta \\ &= \frac{1}{2}i\int_{2\pi}^{0} e^{2i\theta}d\theta \\ &= -\frac{1}{4}(-e^{4i\pi} + e^0) = 0\,.\end{aligned}$$

9.9 Strengthening a theorem on linear fractional transformations

Peter M. Jarvis strengthens a standard result on linear fractional transformations. For complex numbers a, b, c and d, the linear fractional transformation

$$w = f(z) = \frac{az+b}{cz+d}$$

has domain $\mathbf{C}\backslash\{-d/c\}$, with the convention that when $c = 0$ then d is nonzero and the domain is all of $\mathbf{C}$. It is often noted that, if the condition $ad - bc \neq 0$ holds, then $f(z)$ is one-one onto its range. In fact, the conclusion follows from the weaker hypothesis that a and c are not both zero.

Proposition *The linear fractional transformation $f(z)$ is one-one from its domain to its range when at least one of a and c is nonzero.*

Proof If $c = 0$, then $a \neq 0$ and $f(z)$ is a nontrivial linear function. The result holds.

Let $c \neq 0$. Then we multiply both sides of the equation defining w by $cz + d$ and rearrange terms to obtain

$$z(cw - a) = -dw + b \, .$$

Hence

$$z = \frac{-dw + b}{cw - a} \, .$$

Thus, given w in the stated range, one can solve for its preimage z. □

CHAPTER 10

Linear and Modern Algebra

10.1 The all-inclusive span

Ayoub B. Ayoub asked his students to check whether the vector $(2, 4, 1, 5)$ belonged to the span of $\{(1, 1, 1, 1), (1, 0, 1, -2), (1, -3, 1, 1)\}$ in $\mathbf{R}^4$. A student who noticed that the three vectors are pairwise orthogonal, used the dot product to calculate the scalar factors x, y, z in the linear combination

$$(2, 4, 1, 5) = x(1, 1, 1, 1) + y(1, 0, 1, -2) + z(1, -3, 1, 1).$$

Taking the dot product of both sides with $(1, 1, 1, 1)$ yields $x = 3$. Similarly, he got $y = -7/6$ and $z = -1/3$. Then the student claimed that the vector belonged to the subspace. □

However, this is not true, as one can see by noting that the first and third components of the three vectors are the same. Equating components gives an inconsistent system of four equations for the three unknowns x, y, z. The method actually delivers the orthogonal projection of $(2, 4, 1, 5)$ on the span of the three given vectors.

10.2 Rotating a vector

Consider the set A of three-dimensional real vectors (a, b, c) for which $ab + bc + ca = 1$. The set A is not empty, since it contains the vector $(1, 1, 0)$. However, one can interpret the set A as consisting of those vectors (a, b, c) whose scalar product with the vector (b, c, a) is equal to 1.

Now, the vector (b, c, a) is the image of (a, b, c) by the rotation with axis $(1, 1, 1)$ through an angle $4\pi/3$. But this scalar product is also the product of the norms of the two vectors multiplied by the cosine of the angle $4\pi/3$ between them, and this makes it equal to $(a^2 + b^2 + c^2) \cdot (-1/2)$.

Since this scalar product is equal to 1, we get $a^2 + b^2 + c^2 = -2$, and so the set A is in fact empty. □

Thanks to Elliott Cohen for this item.

10.3 Linearly dependent sets of polynomials

Students in the class of R. Bruce Mattingly were asked to explain why the functions $f(x) = 1 - x$, $g(x) = 1 - x^2$ and $h(x) = 3x^2 - 2x - 1$ did not form a linearly independent set. (This was a problem drawn from Exercise 14 of Section 4.5 of *Elementary Linear Algebra*, *4th edition*, by Larson and Edwards, published in 2000 by Houghton Mifflin.)

Several students noted that

$$c_1 f(1) + c_2 g(1) + c_3 h(1) = c_1 \cdot 0 + c_2 \cdot 0 + c_3 \cdot 0 = 0$$

for any choices of scalars c_1, c_2 and c_3, and concluded that the set is linearly dependent. However, is this such an outrageous thing to do? Mattingly reflected on this and realized that the following is true:

Theorem Let $S = \{p_0(x), p_1(x), \cdots, p_n(x)\}$ be a subset of the vector space of all polynomials whose degrees do not exceed n. If the polynomials in S share a common root r, then S is linearly dependent.

To see this, note that, for each i, there is a polynomial $q_i(x)$ of degree $n-1$ or less for which $p_i(x) = (x-r)q_i(x)$ and that the $q_i(x)$ constitute a set of $n+1$ elements (with degrees not exceeding $n-1$) of an n-dimensional vector space, and so a linearly dependent set.

10.4 Correcting Cramer's Rule

Vagarshak Vardanyan points out a misunderstanding about Cramer's Rule for solving a system $AX = B$ of linear equations in intermediate algebra texts, such as [1, 2, 3] as well as Wikipedia [4]. Considering, for example, a system of three linear equations in the unknowns x, y, z, it is contended that if the determinant D of the matrix of coefficients along with the determinants D_x, D_y and D_z of the matrices determined by replacing the appropriate column of the coefficient matrix by column on the right side by B all vanish, then the system has infinitely many solutions.

The following example shows that this is not the case:

$$\begin{aligned} x + 2y + 3z &= 4 \\ x + 2y + 3z &= 5 \\ x + 2y + 3z &= 6 \end{aligned}$$

A more discriminating analysis requires one to consider the ranks of the matrix A and the augmented matrix $A|B$. If these two ranks are equal, there is at least one solution; if the rank of A is strictly less than that of $A|B$, then there is no solution.

References

[1] M.L. Bittinger, *Intermediate Algebra.* Addison Wesley, Boston, 2007.

[2] C.P. McKeague, *Intermediate Algebra.* 8th edition, Brooks-Cole, Belmont, CA.

[3] M. Sullivan and K.R. Struve, *Intermediate Algebra.* 2nd edition, Prentice Hall, Upper Saddle River, NJ, 2010.

[4] Wikipedia, Cramer's Rule. `en.wikipedia.org/wiki/Cramer's_rule` (as of December 1, 2010).

10.5 Evaluating a determinant

Problem Let M be an $n \times n$ matrix for which $M^2 = \alpha M$ for some scalar α. Evaluate the determinant, $\det(I + M)$, of $I + M$, where I is the identity matrix.

Solution (from a student) Since

$$\begin{aligned} M(I + M) &= M + M^2 = (1 + \alpha)M, \\ (\det M)\big(\det(I + M)\big) &= \det\big(M(I + M)\big) \\ &= \det\big((1 + \alpha)M\big) \\ &= (1 + \alpha)^n \det M, \end{aligned}$$

whence $\det(I + M) = (1 + \alpha)^n$. In particular, $\det(I + M) = 1$ when $M^2 = O$. □

Observe that, if

$$M = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

then

$$M^2 = -2M, \quad I + M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\det(I + M) = -1$.

This item is from Michel Bataille, who presents it now regularly to his students, asking them to spot the flaw and solve the problem correctly. The solution is fine as long as M is nonsingular.

10.6 Creating an idempotent

Proposition If A is an idempotent matrix (with $A^2 = A$), then $I + A$ is idempotent as well.

Proof We have that $I - A^2 = (I - A)(I + A)$. Since A is an idempotent, we deduce that $I - A = (I - A)(I + A)$. Multiplying through on the left by $(I - A)^{-1}$ yields $I = I + A$. Finally, multiplying both sides by $I + A$ gives the desired result: $I + A = (I + A)^2$. □

The foregoing argument was given by a student on a mathematics for economics test set by Douglas W. Mitchell.

Comment This proof requires that $I - A$ be nonsingular. However, $(I - A)^2 = I - 2A + A^2 = I - A$, so that $I - A$ is an idempotent and therefore always singular when A is nonzero. Consider, for example, the 2×2 matrix with all entries equal to $\frac{1}{2}$.

10.7 Is this hermitian?

Problem 10377 in the *American Mathematical Monthly* 101 (1994), 362; 104 (1997), 277–278 had a student on a final examination being asked for the definition that A is a hermitian matrix. Instead of the correct result, $A = A^*$, he wrote instead $AA^* = A^2$. Was the answer correct?

The answer is indeed correct, and the cited reference provides five different proofs of this fact.

10.8 Big and little hyperplanes

Elliott Cohen invites us to consider the vector space E of all converging sequences of complex numbers. For such a sequence $x = \{x_n\}$ we define the linear functionals f, g and h:

$$\begin{aligned} f(x) &= \lim_{n\to\infty} x_n \\ g(x) &= \text{Re } f(x) \\ \text{and} \qquad h(x) &= \text{Im } f(x). \end{aligned}$$

It is plain that Kernel f = Kernel $g \cap$ Kernel h; so that we have three distinct hyperplanes, one of which is the intersection of the other two. Another way of looking at it is to note that the hyperplane Kernel f is properly contained in the hyperplane Kernel g.

10.9 Reciprocals in Finite Rings

Michelle Manes teaches an advanced topics course for seniors in Arlington High School in Arlington, MA, that includes material on algebra and cryptography. On a quiz, she required students to determine

(1) the reciprocal of 21 in $\mathbf{Z}/26\mathbf{Z}$;

(2) the reciprocal of 17 in $\mathbf{Z}/29\mathbf{Z}$.

The numbers were selected because they allowed relatively short computations using the Euclidean algorithm, yielding positive numbers. She did not anticipate that "almost everything students did produced the right answers."

One student said that $26 \equiv 5 \pmod{21}$ and $29 \equiv 12 \pmod{17}$, so that the correct answers to (1) and (2) were respectively 5 and 12. Another student misinterpreted Fermat's Little Theorem to provide that $a^{a-1} \equiv 1 \pmod{p}$. Since $21^{19} \equiv 5 \pmod{26}$ and $17^{15} \equiv 12 \pmod{29}$, the respective answers are again 5 and 12. More mysterious was the student who noted that $17^{11} \equiv 12 \pmod{29}$, so the answer to (2) is 12 (where did the 11 come from?).

In both (1) and (2), the multiplicative and additive inverses are equal. This might have thrown off some students who in fact solved the problem correctly, but worried that they had made a mistake.

10.10 If it's in the Textbook, it must be true.

The following note is by Donald A. Teats, of the South Dakota School of Mines and Technology in Rapid City, SD.

A student was in my office recently questioning why I had marked a problem wrong on his numerical analysis examination. I explained to him the commonly used convergence criterion for the Gauss-Seidel method for solving $\mathbf{Ax = b}$: the method converges if the matrix $\mathbf{A}$ is *diagonally dominant*, *i.e.*, if the absolute value of the diagonal entry in each row is greater than the sum of the absolute values of the off-diagonal entries in that row. To show that his claim of "greater than or equal to" in the description of diagonal dominance was incorrect, I demonstrated that the Gauss-Seidel method fails in the solutions of $\mathbf{Ax = b}$ with

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

unless one is fortunate enough to get the second and third coordinates exactly right in the starting vector.

Later in the course, the Gauss-Seidel method came up again, this time in our study of numerical solutions to Lapace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Using the centered difference approximation

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+h, y) - 2u(x, y) + u(x-h, y)}{h^2}$$

and its analogue for $\frac{\partial^2 u}{\partial y^2}$, Laplace's equation can be replaced by

$$u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h) - 4u(x, y) = 0.$$

Denoting $u(x, y)$ by u_{ij}, $u(x+h, y)$ by $u_{i+1,j}$, etc., this can be rewritten as

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} = 0. \qquad (*)$$

We can approximate u on a rectangular region $a \le x \le b$, $c \le y \le d$ at mn gridpoints $(a + ih, c + jh)$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots n$ by solving a sparse linear system of mn equations in mn unknowns, each equation having the form $(*)$. (Boundary values $u_{0,j}$, $u_{m+1,j}$, $u_{i,0}$, $u_{i,n+1}$ are assumed to be known.) For example, if $m = n = 3$, the coefficient matrix for the linear system is

$$\begin{pmatrix}
-4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4
\end{pmatrix}$$

The solving of this system by the Gauss-Seidel method is known as *Liebmann's method*. When I presented this method to my class, my very astute student brought to my attention the textbook's claim that "because [the matrix defined by $(*)$] is diagonally dominant, this procedure will eventually converge on a stable solution" [2, p. 817], .

"Hmm," I thought, "this matrix isn't diagonally dominant," and I did what anyone might do under similar circumstances to quickly prove the point. I pulled another book [3] off the shelf and turned to the section on numerical

solutions of PDE's, and to my amazement, I found the same result: the coefficient matrix for the system $(*)$ "is clearly diagonally dominant since in each row $|-4|$ is greater than the sum of the absolute values of the other entries." Greater than? No! Greater than or equal to? Yes, but that is not enough for the matrix to be diagonally dominant. As long as there are textbooks there will be errors in textbooks, but to find an error that has persisted through several editions of two widely used textbooks is a bit unusual!

So does Liebmann's method converge or not? Indeed it does, but not for the reason claimed by these two textbooks. In [1, p. 560], one finds the correct analysis. First, multiply equation $(*)$ by -1, so that the resulting matrix $\mathbf{A}$ has 4's on the main diagonal, with all other entries either 0 or -1. The matrix $\mathbf{A}$ is symmetric; thus its eigenvalues are real. One can use the Gerschgorin theorem and the fact that $\mathbf{A}$ is nonsingular to show that all the eigenvalues are positive, so that $\mathbf{A}$ is positive definite. And that, rather than diagonal dominance, is what gives the convergence of the Gauss-Seidel method. In [1, p. 551], , we find the following theorem: "Let $\mathbf{A}$ be Hermitian with positive definite elements. Then the Gauss-Seidel method for solving $\mathbf{Ax = b}$ will converge, for any choice of $\mathbf{x}_0$, if and only if $\mathbf{A}$ is positive definite."

My student was convinced. Are you?

References

[1] Kendell E. Atkinson, *An Introduction to Numerical Analysis*, 2nd edition. John Wiley and Sons, 1989.

[2] Steven C. Chapra & Raymond P. Canale, *Numerical Methods for Engineers*, 3rd edition. McGraw-Hill, 1998.

[3] Dennis G. Zill & Michael R. Cullen, *Differential Equations with Boundary-Value Problems*, 4th edition. Brooks-Cole, 1997.

10.11 Matrix inverses and the great injustice

Zoran Sunik of the University of Nebraska, Lincoln, NE created this plea from an aggrieved "student."

Dear Editor, I am writing this letter to you in the hope that you or your readers can help me with this matter. A great injustice has been done and I would appreciate any help that I can get. I desperately need to get an A in this course in order to bump up my GPA enough to get accepted into my

major. Unfortunately, the teacher says he cannot give me anything better than A−. But, let me explain first what happened.

A couple of days ago, the following problem was given to us on the test.

Problem Determine the inverse matrix of

$$M = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}.$$

Now this was a very unfair question. You see, we all expected to be given a 3×3 matrix. I practiced and practiced on the 3×3 case and I remember everything so well. You cross the rows, you cross the columns, calculate the determinants of the remaining parts, switch the signs according to the checkerboard pattern, you flip the whole thing and you are done. When he was teaching, the teacher used some fancy words like minors, factors, adjuncts and whatnot, but believe me, what I said before is all you have to do to get the inverse right. Oh yes, divide by the determinant at the end.

If you are confused by my explanations, you can check what my textbook says about this method [1, Section 3.3].

I was sitting there and I tried to remember how to do the 2×2 case in a similar fashion. However, if I cross a row and a column all that is left is one number and I certainly cannot calculate determinants when there is no matrix.

So I tried another thing. I remembered that the inverse matrix must be a matrix

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

of size 2×2 such that

$$MX = I,$$

where I is the indemnity matrix. When I multiplied the left-hand side I got

$$\begin{pmatrix} x_1 + 3x_3 & x_2 + 3x_4 \\ x_1 + x_3 & x_2 + x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and when I set the corresponding entries equal I obtained the following system

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right).$$

At this point I again realized that I was in trouble. I knew there was this messy method of solving any system by somehow making some entries 1 and some 0, but it was too complicated and I never quite understood which entries had to be made 1 and which 0. The teacher was probably unsure too, because he made some entries 1 in some problems and made them 0 in others. This is supposedly explained in the very first chapter of my textbook [1, Section 1.2], and my roommate claims that she actually got it from there. But she claims so many things.

Then I remembered Cramer's Rule. I named the 4×4 matrix of the system by A and I knew that I needed to calculate $\det(A)$. Oh, no! How do I calculate a 4×4 determinant? The teacher said we were not going to be given 4×4 determinants on the test; we only had to know the 3×3 and 2×2 cases. Then I said to myself, why not use the same method that works for 3×3? I rewrote the first three columns of A to the right of A, calculated the four products going down to the right with plus sign

$$\begin{pmatrix}
+ & & + & & + & & + & & & & & \\
 & 1 & & 0 & & 3 & & 0 & & 1 & & 0 & & 3 \\
 & & \diagdown & & \diagdown & & \diagdown & & \diagdown & & & & & \\
 & 0 & & 1 & & 0 & & 3 & & 0 & & 1 & & 0 \\
 & & & \diagdown & & \diagdown & & \diagdown & & \diagdown & & & & \\
 & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 \\
 & & & & & \diagdown & & \diagdown & & \diagdown & & \diagdown & & \\
 & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0
\end{pmatrix}$$

and the four products going down and to the left with minus sign

$$\begin{pmatrix}
 & & & & & & - & & - & & - & & - & \\
1 & & 0 & & 3 & & 0 & & 1 & & 0 & & 3 & \\
 & & & & & \diagup & & \diagup & & \diagup & & \diagup & & \\
0 & & 1 & & 0 & & 3 & & 0 & & 1 & & 0 & \\
 & & & \diagup & & \diagup & & \diagup & & \diagup & & & & \\
1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & \\
 & \diagup & & \diagup & & \diagup & & \diagup & & & & & & \\
0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 &
\end{pmatrix}$$

and I obtained $\det(A) = 1 + 0 + 9 + 0 - 3 - 0 - 3 - 0 = 4$. In exactly the same fashion I calculated the other four determinants that I needed for the Cramer's Rule, namely the determinants $\det(A_1) = -2$, $\det(A_2) = 6$,

$\det(A_3) = 2$ and $\det(A_4) = -2$. Then I calculated

$$x_1 = \frac{\det(A_1)}{\det(A)} = -\frac{1}{2}$$
$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{3}{2}$$
$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{1}{2}$$
$$x_4 = \frac{\det(A_4)}{\det(A)} = -\frac{1}{2}$$

and concluded that

$$M^{-1} = X = \begin{pmatrix} -1/2 & 3/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

I was very tired by the end of these inextricable calculations, but I managed to convince myself to check the result at the end, mainly because I heard that the teacher usually gave partial credit for almost anything the students wrote down. And there it was! It checked out nicely. I actually solved the problem; I was actually good in math after all. I knew that all the time, but I had very bad teachers before and it took me a while to realize that ... Oh, where was I?

Anyway, let me go back to my case. The teacher gave me 7 out of 10 points for my solution. I complained and complained, of course. I had a correct solution. I explained and explained to my teacher that everything I did was good, but he did not want to listen. He said that, for the most part, he was very happy with my work, but I still did not deserve the full credit. For example, he did not like the fact that I did not know the simple way to do the problem, but he liked that, rather than remembering formulae, I actually knew what an inverse matrix was. Introducing unknowns and setting a system to find them was a lovely idea, he said. I should have used the row eschalot matrix to solve the system though, he said, and he was not a big fan of Cramer's Rule either. Nevertheless, he was delighted that I used it properly, because he never mentioned that it works for more than three unknowns. Then, he shook his head and said that it was unfortunate that I did not know how to calculate determinants by expansion, and I still have no clue what he meant by that. He concluded by trying to convince me that the method I used to calculate determinants was not good at all. I was just lucky to get the correct answer, he said. The trick with the plus products going to the right and minus products going to the left worked only with

matrices of order 3 (it took me a while to realize that that's his fancy way of calling the 3×3 matrices). He even showed me an example of a 4×4 matrix for which he claimed that the trick did not work. He had me right there; what could I say? I did not know how to calculate determinants by his expansions and I had to believe him.

I went home, watched TV with my roommate and tried to forget the whole thing. My roommate is a very annoying person, and she kept asking me about the exam. She is a math major and she actually enjoys talking math so she insisted and insisted, until I told her about it. I did not have the test with me; I threw it as soon as I left the teacher's office, of course, and I just put some random numbers for the entries in the matrix M, to explain what I did. Sure enough, all these math people are the same. As soon as I explained how I calculated the 4×4 determinants she screamed, "But no, you cannot do that." When I asked why not, she said, "Finish the problem just like you did in your test and check the matrix at the end. You were lucky with the matrix on the test, but since this is a different matrix it is very unlikely you will be lucky again." I calculated everything for this new matrix and checked the answer. And there it was! It worked! She did not believe me and she checked everything, but she got the same result. Then she tried different and different examples and my method always gave the correct answer. Always! For all examples we tried.

My roommate tried to discourage me yesterday by showing me an example of a 3×3 matrix M for which my method of matrix inversion failed. She introduced a matrix X with nine unknowns, got a 9×9 system, calculated the needed determinants by extending the matrix of the system by its first 8 columns and then using the 9 products going down and to the right with the plus sign and the 9 products going down and to the left with the minus sign, and finally she used the Cramer's Rule to get the entries of X, but the final answer did not check out.

Despite all this, she never found an example of a 2×2 matrix for which my method failed. So I am asking you, does my method always give the correct inverse matrix in the 2×2 case? And if it does, then it is a correct method, isn't it? Why did the teacher take off my three points then? Let me remind you that all this is not just an academic discussion. Without the three points, I am out of the A range in the course and I cannot get into my major. This ticks me off so much and I became so interested and involved that all I could do for the last three days was think about it and try to prove that my method works.

What do you think?

Zoran

Reference

[1] David Lay, *Linear Algebra and its Applications. Second edition.* Addison Wesley, 1996.

Answer from the editor Dear Zoran, your method presents a startling, idiosyncratic, convoluted and perfectly valid way to invert a matrix of order 2 (I prefer the fancy way too). Still, your teacher treated and graded you fairly. Rather than explaining how you can both be right, let me move on to the important thing. Forget about the three points. Your letter shows some of your weaknesses (indemnity matrix? eschalot form? inability to perform row reduction! *etc.*), but it also shows signs of understanding, persistence, curiosity and brilliance that would lead you to a very successful career in mathematics. Go for it!

CHAPTER 11

Miscellaneous

11.1 Unexpected gaps

Sung Soo Kim recalls his puzzlement when he first learned of the countability of the rationals and the uncountability of the irrationals. In view of the density of both rationals and irrationals in the reals, he wondered how an uncountable number of gaps could be produced from only a countable number of points while only a countable number of gaps could be generated from an uncountable number of separated points.

11.2 Measuring humor

The front page of the June 15, 2004, edition of the *National Post* in Toronto carried the account of a formula to measure the effectiveness of jokes. According to the report:

> The formula is
>
> $$x = \frac{(f\,l + n^o)}{p}$$
>
> where f refers to the funniness of the punchline, a factor that is multiplied by l or length of the buildup. That total is added to n — which represents how much the audience doubles over. The n factor has an exponent of o referring to the 'ouch element' or degree of human pain and embarrassment. All of this is divided by p or the number of puns, which are known among comedic circles to reduce the degree of funniness. Each factor has a maximum score of ten.
>
> The formula's creators, Timandra Harkness and Helen Pilcher — a stand-up comic and a neuroscientist — are behind *The Comedy*

Research Project in Britain, which aims to prove that science can be funny.

Putting aside the derogatory opinion of puns, which in my experience seems to be a preferred type of humor among many mathematicians, I find this to be another attempt to lend authority to a dodgy bit of business through the invocation of something that looks mathematical. Four examples of jokes are provided with their evaluations. The world's funniest joke, according to Laughlab at the University of Hertfordshire, is the following.

> Two hunters are out in the woods when one of them collapses. He doesn't seem to be breathing and his eyes are glazed. The other guy takes out his phone and calls emergency services. He gasps, "My friend is dead! What can I do?" The operator says, "Calm down; I can help. First, let's make sure he's dead." There is silence; then a gunshot is heard. Back on the phone, the guy says: "OK. Now what?"

The Humor Quotient of this is evaluated as (*sic!*)

$$x = \frac{(7 \times 9 + 0)}{0} = 63.$$

Three other jokes with their evaluations follow.

> A man walks into a Chinese restaurant. In the lobby are two large lobster tanks, one with a cover, one without. "I see that you have lobster," says the man to the maitre-d'. "Oh yes," the maitre-d' replies. "Why," asks the man "are they in two different tanks?" The maitre-d' explains that one tank contains lobsters caught in American waters — the one with the cover on — while the other is for lobsters caught in Canadian waters. "Why," the man asks, "does the American lobster tank have a lid?" The maitre-d' explains that this is because they try to climb out. "But doesn't the Canadian lobster do the same?" the man asks. "Yes," the maitre-d' replies, "but when the Canadian lobster climbs out, the others pull it back in."

Humor quotient: $x = (9 \times 8 + 5^2)/0 = 97$.

> A guy walks up to his house and sees a snail sitting on his porch, which he boots away. A year later, his doorbell rings and he opens the door to find a snail sitting there. "Hey," says the snail, "what's with the kick?"

Humor quotient: $x = (10 \times 7 + 0)/0 = 70$.

Q: What is brown and sticky?

A: A stick.

Humor quotient: $x = (8 \times 1 + 0)/0 = 8$.

11.3 Braess' Paradox

At a Joint Mathematics Meeting in Phoenix, AZ, Eva Tardos of Cornell University in Ithaca, NY, gave an interesting talk on network games and approximation algebras. In the course of this, she referred to the following paradox.

Suppose that we have a stream of traffic that must travel from S (Start) to F (Finish); two roads are available, one passes through a point A and the second passes through a point B. Initially, one can pass from one of these roads to the other only at S and F. Regardless of the amount of traffic, it takes each vehicle one hour to travel from S to B, and one hour to travel from A to F. As for the other legs, the amount of time taken to travel is proportional to the amount of traffic. If the fraction x of the vehicles takes that leg rather than the alternative, then we suppose that each vehicle requires x hours.

Thus, if x of the traffic follows the route $S-A-F$ and $1-x$ of the traffic follows the route $S-B-F$, then each vehicle on the former route needs $x+1$ hours to complete the journey and each on the latter route requires $1+(1-x) = 2-x$. If $x < \frac{1}{2}$, then vehicles starting out on the $S-B-F$ route have an incentive to change their plans in order to save a little bit of time. However, when $x = \frac{1}{2}$, then everyone requires one and a half hours, and nobody is tempted to change their mind; thus this situation has the stability of a Nash equilibrium in that no one can improve his position by a unilateral change of plan.

Now, suppose that we connect A and B by a road that requires no time to traverse. Then everyone will benefit, or at least do no worse, by starting the journey by going from S to A; the time required will be at most one hour. However, once at A, if everyone seeks to minimize his time, then all will want to go to B and finish the journey along the $B-F$ leg. Thus, even though we have provided more alternatives for the traffic flow, if everyone behaves selfishly by taking the $S-A-B-F$ route, then it will take two hours for everyone to complete the journey, a half hour longer than in the more restricted situation.

11.4 The Limits of Reason

Andrea Rothbart sent in an item that appeared in *Newsweek* on August 5, 2010. In a piece entitled *The Limits of Reason; Why evolution may favor irrationality*, Sharon Begley, a postdoctoral fellow in the Philosophy, Politics and Economics program at the University of Pennsylvania, wrote:

> Another form of flawed reasoning shows up in logical puzzles. Consider the syllogism "No C are B; all B are A; therefore some A are not C." Is it true? Fewer than 10 percent of us figure out that it is, says Mercier. One reason is that to evaluate its validity requires constructing counterexamples (finding an A that is a C, for instance).

11.5 Particle in circular motion

Peter M. Jarvis sends in the following counterpoint between the physical and mathematical approaches to common problems.

Problem A particle undergoes uniform motion on a circle of radius R with a period of T seconds. Determine its speed.

Mathematician's solution Since the motion is uniform, the particle goes through 2π radians in T seconds, so its location as a function of time (assuming the center of the circle is at the origin) is

$$L(t) = R\left[\cos\left(\frac{2\pi}{T}t\right)\hat{\mathbf{x}} + \sin\left(\frac{2\pi}{T}t\right)\hat{\mathbf{y}}\right].$$

Differentiating and taking the magnitude yields

$$\begin{aligned}|L'(t)| &= \sqrt{\left(R\frac{2\pi}{T}\right)^2 \sin^2\left(\frac{2\pi}{T}t\right) + \left(R\frac{2\pi}{T}\right)^2 \cos^2\left(\frac{2\pi}{T}t\right)} \\ &= \frac{R2\pi}{T}.\end{aligned}$$

Physicist's solution Since the motion is uniform, the speed is constant and equal to the circumference divided by the time taken, namely $2\pi R/T$.

Problem In the foregoing situation, determine the acceleration of the particle.

Mathematician's solution Differentiate $L(t)$ twice and take the magnitude to get acceleration of constant magnitude

$$|L''(t)| = \sqrt{R^2\left(\frac{2\pi}{T}\right)^4 \cos^2\left(\frac{2\pi}{T}t\right) + R^2\left(\frac{2\pi}{T}\right)^4 \sin^2\left(\frac{2\pi}{T}t\right)}$$
$$= \frac{4\pi^2 R}{T^2}.$$

Physicist's solution By symmetry, the magnitude of the acceleration must be constant. When the particle is on the 'rightmost' point of the circle, its speed is $2\pi R/T$ and pointed up. When the particle is on the 'leftmost' point of the circle (a half revolution later), it has the same speed, but is now pointed down. Since the acceleration is constant, it is the change in speed divided by the time taken:

$$\frac{(2\pi R)/T - (-2\pi R)/T}{\frac{1}{2}T} = \frac{8\pi R}{T^2}.$$

11.6 The Plucked String

The following essay due to Tommaso Toffoli of Boston University.

Consider a piece of elastic string stretched between two fixed points, as in Figure 1(a). This is a piece of ideal *harmonic string*, i.e., its mass is uni-

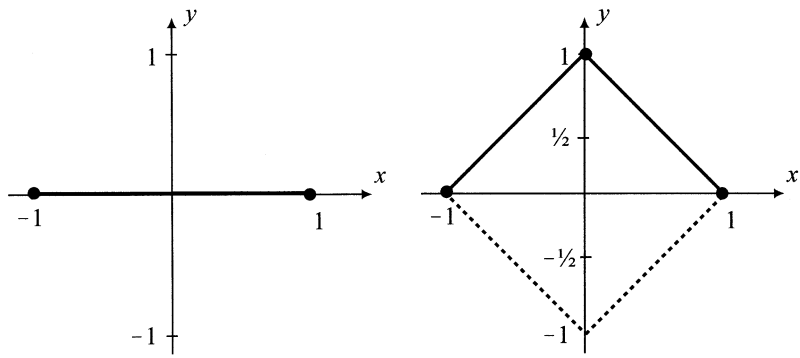

Figure 1. (a) An elastic string (solid line) is stretched between endpoints -1 and $+1$. (b) If the string is plucked at midpoint as indicated, it will swing between this starting configuration and its mirror image (dotted line).

formly distributed along x and the restoring force is strictly proportional to the displacement along y. (Actually, all we need is *linearity*, which even with a less-than-ideal string we can approximate as closely as we wish by restricting our attention to sufficiently small oscillations.) As is well known (see, e.g., Frank Crawford: *Waves*, Berkeley Physics Course, vol. 3, McGraw–Hill 1968) the spacetime evolution $y(x,t)$ of such a string obeys the *wave equation*

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

where c may be called the *speed of sound*; any solution of this equation can be viewed as the superposition of two waveshapes, traveling one rightwards and the other leftwards at the speed of sound (and appropriately reflected at the endpoints).

If this string is "plucked" at midpoint, that is, pulled into the static configuration of Figure 1(b) (solid line) and let go, it will swing back and forth between this configuration and its mirror image (dotted line), completing a cycle in a time T. Correspondingly, its center of mass will oscillate along the y-axis between $\frac{1}{2}$ and $-\frac{1}{2}$, its position being given by the function $g(t)$ plotted in Figure 2. (The string is plucked at time $t = 0$. Units are chosen so that $c = 1$, and thus $T = 4$.)

Question: *Just what kind of function is $g(t)$?*

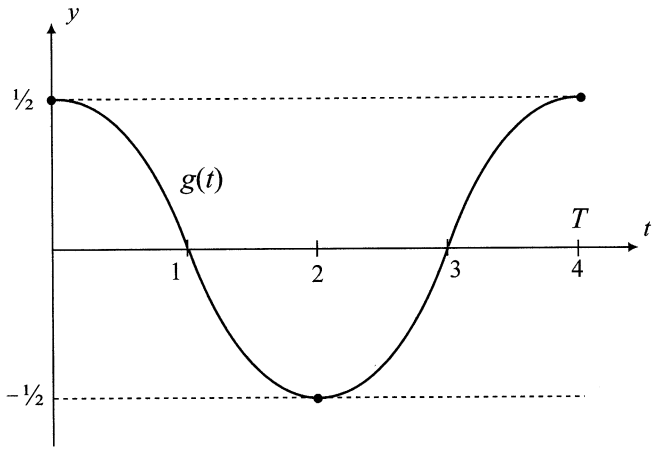

Figure 2. As the string swings between the two configuration of Figure 1(b), its center of mass oscillates between points $\frac{1}{2}$ and $-\frac{1}{2}$. Here we plot its position as a function of time, $g(t)$.

Answer. Did you guess that

$$g(t) = \frac{1}{2}\cos 2\pi \frac{t}{T}?$$

Try again! The curve $g(t)$ is not a *trigonometric* function, but a composite of *parabolic arcs*, as indicated in Figure 3.

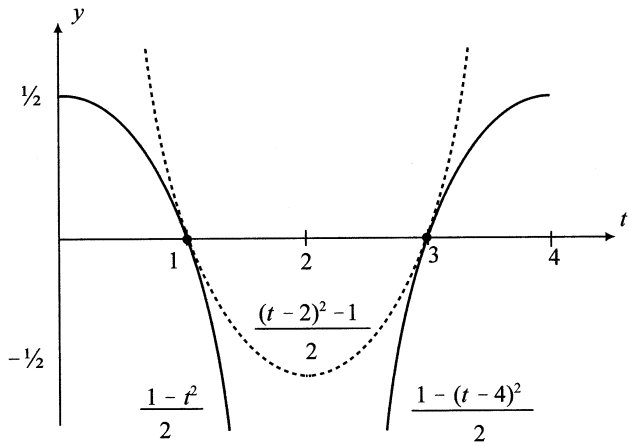

Figure 3. The function $g(t)$ is not a cosine but a composite of arcs of parabola.

To see this, one might be tempted to integrate the wave equation from the given initial conditions (using, for instance, the Fourier transform), and from that, by averaging over the length of the string, eventually obtain the trajectory of the center of mass. In what follows, instead, we directly seek just the motion of the center of mass (but nonetheless we will get a complete description of the string's motion as a byproduct.)

On the harmonic string, disturbances travel along x at exactly the speed of sound $c\ (= 1)$—nothing faster. At any time $0 < t < 1$, the information that the midpoint has been released will have spread right and left from this point to the ends of the interval $S = [-t, t]$—which we may call the *sound cone*; as a consequence, the portion of string comprised within the sound cone—the *inner* portion—may be found to have moved from the initial position (Figure 4a). On the other hand, until time t the portion of string outside the sound cone—the *outer* portion—will not yet have "felt" that the string has been released, and consequently (same figure) *cannot* have moved! During the entire time interval $(0, 1)$ the force pulling the string downwards from the two endpoints remains *constant*, and thus the center of mass describes a *parabola* in the t-y plane.

It remains to be shown that during subsequent time units this behavior simply goes through alternate reflections.

Note that inner and outer portions of the string are not *fixed* pieces of string material that are being stretched or shrunk as t progresses; rather, it is the *boundary* between inner and outer portions—the edge of the sound cone—that moves outwards, encompassing an increasing fraction of the string.

Now, the inner portion of the string consists of a *linearly* growing amount of string material, of length $2t$ and thus of mass mass $m(t) = 2\mu t$ (where μ is the uniform mass density). If we call $v(t)$ the downward velocity of the center of mass of this inner portion, its momentum at time t will be $p(t) = 2\mu t\, v(t)$.

The outer portion, for its part, consists of two pieces, anchored to the endpoints, having constant slope and thus constant stress. Thus, these two pieces will have been pulling the inner portion downwards, each with a *constant* force f, and thus by time t they will have transfered to that portion a momentum $p(t) = 2ft$.

By the action–reaction principle, the momentum imparted by the force must equal that acquired by the inner portion of the string; thence

$$2\mu t\, v(t) = 2ft, \quad \text{or} \quad v(t) = f/\mu = \text{const} = 1$$

(the value 1 for the constant derives from our choice of units). Thus, the inner portion's center of mass moves downwards at a *constant velocity*. Incidentally, since this holds from the very first instant, the shape of the string at time t must actually be that of Figure 4b, with the inner part proceeding downwards on a *straight front*. In other words, the way the inner part gains

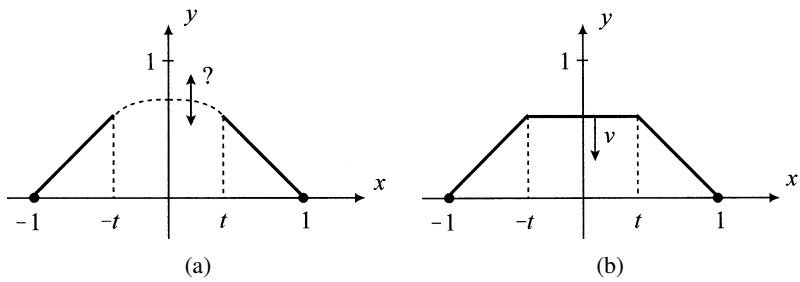

Figure 4. (a) Releasing the string from the shape of Figure 1(b) generates a disturbance that propagates outwards at the speed of sound from $x = 0$, and by time t will have affected the interval $[-t, t]$ (inner region of the "sound cone")—but not what lies outside it. (b) A bit of thought (see text) shows that within this inner region the string must be moving in a *straight* front with a *constant* velocity v.

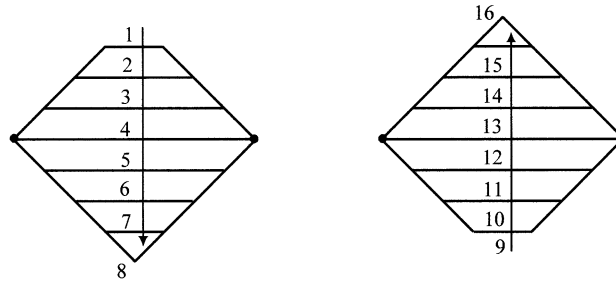

Figure 5. Sequence of string configurations and direction of motion for one cycle, in increments of $1/16$ of a cycle.

momentum is not by moving faster but by accreting new material at the edges. The overall position of the center of mass comes out as the weighted mean of those of inner and outer portions:

$$\underbrace{t \times (1-t)}_{\text{inner, weight} \times \text{height}} + \underbrace{(1-t) \times \tfrac{1}{2}(1-t)}_{\text{outer, weight} \times \text{height}} = \frac{1-t^2}{2}.$$

At time 1 the sound cone will have advanced over the entire string, which at this instant will run straight from one endpoint to the other (frame 4 in Figure 5), the entire front traveling downwards. As soon as the string passes to the other side of the x-axis, the endpoints will start pulling *upwards* rather than downwards, tending to retard the string's motion. In fact, the moving region of the string, while still proceeding downwards at unit speed, will gradually shrink in extent until, at time 2, it will have contracted to a point (frame 8 in Figure 5). The motion will proceed in reverse through the second half of the cycle. The whole sequence of string configurations is summarized in Figure 5.

11.7 Can the Goldbach conjecture be undecidable?

CHANCE News 9.08 for July 3, 2000, to August 9, 2000, draws attention to an article by Jim Holt in *Lingua Franca* (July/August, 2000) `www.linguafranca.com/0007/hypo.html` in which the author makes the following comment:

> From Gödel's incompleteness theorem we know that in any formal system of arithmetic there are infinitely many propositions that are neither provable nor disprovable. Could Goldbach's conjecture be one of them? (That is what Uncle Petros [in the book, *Uncle Petros and*

> *Goldbach's Conjecture* by A.K. Doxiadis] begins to suspect.) If Goldbach's conjecture could be shown to be undecidable — neither provable nor disprovable — then this would be tantamount to proving it true! For if it was false, there must be some counterexample to it. But such a counterexample would constitute a disproof of the conjecture — thereby contradicting its undecidability.

The Goldbach conjecture is that every even number exceeding 4 is the sum of two primes.

11.8 Do contradictions exist?

Theodore G. Ammon, Department of Philosophy, Millsaps College, Jackson, MS, argues in the following paragraphs that there are no contradictions.

A contradiction either implicitly or explicitly asserts both a claim and the claim's denial. In logical notation, a contradiction might be rendered as '$P \& \sim P$'. Therefore to assert that '$\sim (P \& \sim P)$' is simply to demand consistency in a formal system such as logic. But since the concept of a contradiction is indifferent to the type of formal system, one can declare the principle of non-contradiction as an axiom, or, as Aristotle does, declare it to be a law of thought. No matter, the bottom line is that in formal systems such as mathematics and logic, whether axiomatized or not, we bank on the assumption of consistency and hence on the assumption that contradictions are not allowed as true. Hence, to derive a contradiction from a set of assumptions suffices to prove at least one of the assumptions false.

We can thus use the presence of a contradiction as an indicator that something is awry, and by indirection, prove what **is** the case by proving what **isn't**. In a proof in symbolic logic, for example, if the denial of the conclusion leads to the absurdity of a contradiction then we know that the conclusion as stated must be the case. Ditto in mathematics.

Consider now the concept of validity, while keeping in mind what a contradiction is. If a proof is valid then the premises guarantee the conclusion. Put differently: if a proof is valid then it is impossible for the premises to be true and the conclusion false. But if a set of propositions contains a contradiction then it is impossible for the premises to be true and the conclusion false, since the contradiction will always be false. Given a contradiction, any proposition will follow. As a tautology is implied by any proposition, a contradiction implies every proposition. Thus any proof must be valid if the propositions of the premises are contradictory. Consider the following arguments:

1. $P \& \sim P \ / \ \therefore Z$
2. $P \& \sim P \ / \ \therefore \sim Z$.

Both arguments are valid and are easily proven valid by truth-tables or traditional rules of inference.

But now consider arguments cast in the form of traditional syllogisms. In Categorical Logic there are four propositions, and even in the post-Boolean interpretation of the syllogisms, the relationship of "contradictoriness" is preserved. Thus the propositions "All S are P" and "Some S are not P" are contradictories, and the propositions "No S are P" and "Some S are P" are contradictories. In order to construct a syllogism with contradictory premises one would have to have an argument of one of the following forms:

1. All S are P.
2. Some S are not P.

$\therefore$ conclusion.

1. No S are P.
2. Some S are P.

$\therefore$ conclusion.

In syllogistic logic, however, an argument must have three terms and one of them is the so-called "middle term," the one that the premises share. Given the rules that govern the construction of syllogisms, it is impossible to construct an argument that resembles a syllogism that really is a syllogism if the premises are contradictory. Furthermore, neither set of premises above can logically entail a conclusion. If either set of premises were used in an argument, one would have to conclude that the argument is invalid because there would be no middle term, and/or the conclusion would contain a term or terms that do not appear in the premises, another violation of the rules covering syllogisms.

Since the very concept of a contradiction requires that a contradictory set of claims entails any conclusion whatsoever, and since in categorical logic it is impossible for contradictory premises to yield any conclusion at all, it follows that the concept of "contradiction" is itself contradictory.

Hence there are no contradictions.

This essay drew the following comment from Calvin Jongma of Dordt College in Sioux Center, IA.

In the article, *There are no contradictions*, there really are no contradictions. Just a paradoxical conclusion generated by a confusion regarding what constitutes a proof or an argument. This confusion results from blurring the distinction between logical implication and deducibility. These are notions mathematical logicians have adequately distinguished for most of the twentieth century, but which many working mathematicians and students of mathematics may be less familiar with.

To explain the hoax further (that Aristotelian logic cannot generate any results from contradictory sentences), we will distinguish two senses of "argument." In one sense, an argument is merely a list of premises with a conclusion that logically follows from them. We will call this a premise-conclusion argument. Such a premise-conclusion argument is valid iff the premises logically entail the conclusion, *i.e.*, iff it is impossible for the conclusion to be false while the premises are true. Here implication is the issue, not deducibility.

However, we usually mean by an argument something more than the bare-bones premise-conclusion sequence. In the ordinary sense of the term, an argument is a sequence of statements in which the conclusion is actually deduced from the premises in a step by step fashion. We will call this more normal sense of argument a proof or deduction. In a formal logical setting, each line of a deduction must be generated via stipulated rules of inference, each of which is sound (preserves truth). The nature and conclusive character of a deduction is determined both by the particular logic being used and also by the deduction system (set of inference rules) that are chosen for deriving conclusions from premises. In this second sense of an argument, genuine arguing must take place to demonstrate the logical connection holding between the premises and the conclusion. The former notion of a premise-conclusion argument does not involve any argumentation, only logical relations among sentences and connections between their potential truth values. Premise-conclusion arguments are exactly what require proof. A deduction of a conclusion from its premises using sound rules of inference is the way mathematicians establish validity.

Now, in the premise-conclusion sense of argument, where logical consequences are the issue rather than deductive consequences, contradictory categorical sentences X and X_{op} in Aristotelian Logic obviously imply any statement Z that can be stated in the language, just as is the case for Sentential or Propositional Logic. Every time X and X_{op} are both true (never), so is Z. The assertion in the article that they do not, therefore, is simply false.

It is true, though, that one cannot deduce anything from a syllogism

whose premises are contradictories, for the simple reason that, strictly speaking, syllogistic reasoning requires three distinct terms. (If this requirement is relaxed, however, so that only two distinct terms occur, at least the contradictory argument "Some S are not S" can be deduced using the Aristotelian syllogistic form $AOO-2$ on the premises X: "All S are P" and X_{op}: "Some S are not P," even if an arbitrary categorical sentence Z cannot be generated.) This conclusion about the limitations of strictly syllogistic reasoning should not surprise us; nothing much can be deduced from $P\&{\sim}P$ in Sentential Logic either if the rules of inference are restricted to, say, the basic rules covering conditionals. Proving Z and ${\sim}Z$ from $P\&{\sim}P$ requires rules for negation and conjunction. Similarly, in Aristotelian Logic, a Reductio Ad Absurdum rule based on the Square of Opposition can be introduced, and then any categorical conclusion Z and its contradictory Z_{op} can be deduced from the given premises. This is not a syllogistic rule, but then neither are the standard immediate inference rules known as Conversion and Subalternation, which are required to complete the deduction system for Aristotelian Logic.

So if we think about arguments in terms of logical implication, contradictories do imply anything whatsoever, while if we think about them in terms of deductions, they fail to deduce anything only if the rules fail to handle proofs by contradiction. This is true for both Sentential Logic and Aristotelian Logic. Contradictions are fully as constructive in a robust version of Aristotelian Logic as they are in Sentential Logic. Where Aristotelian Logic is limited is in its range of expressibility, not in its ability to imply or deduce logical consequences.

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